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# TRANSITION KERNELS AND DYNAMICAL OPERATORS

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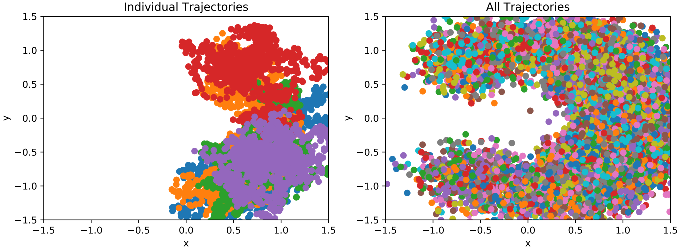
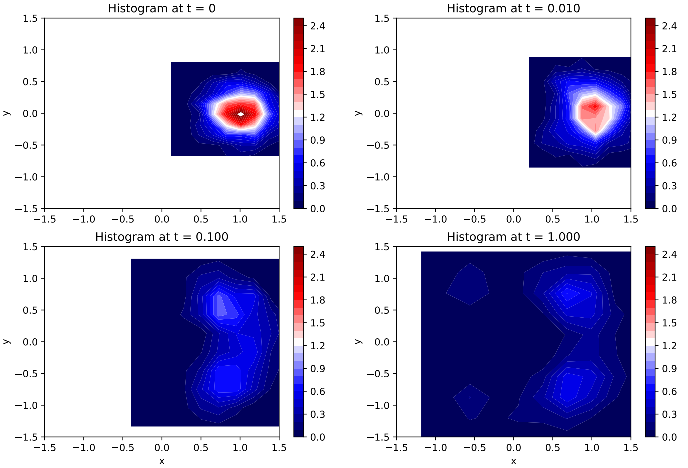
# Trajectories vs Statistics

Example: 1000 realizations of a random process.




Study of individual trajectories is often not too helpful!

# Time Evolution of Statistics



Motivation

Transition Kernels

Dynamical Operators

Stationarity and Reversibility

## Basic Setting

We study a *stochastic process* $X_t$, which assumes a position $X_t \in \mathbb{X}$ in some state space $\mathbb{X}$ for every moment in time $t \geq 0$.

The process is not assumed to be deterministic, but can be stochastic. More formally, there is a probability space $(\Omega, \Sigma, \mathbb{P})$ such that $X_t$ is a family of random variables parametrized by time, i.e. $X_t = X(t, \omega)$ for $\omega \in \Omega$.

## Markov Property

We focus our attention on processes "without memory of the past":

### Definition

A process $X_t$ is a *Markov process* if for all choices of times $s_1 < \ldots < s_l < t$, we have

$$\mathbb{P}(X_t \in A | X_{s_1}, \ldots, X_{s_l}) = \mathbb{P}(X_t \in A | X_{s_l}),$$

"Given knowledge of the present, the future is independent of the past."

## Transition Kernel

Centerpiece for the definition of a Markov process is a *stochastic transition kernel*:

### Definition

A family of two-argument functions $p_\tau = p_\tau(x, A) \in [0, 1]$ is called a family of stochastic transition kernels, if

1. For each $x \in \mathbb{X}$, the map $p_\tau(x, \cdot)$ is a probability measure on $\mathbb{X}$.
2. For each (suitable) $A \subset \mathbb{X}$, the map $p_\tau(\cdot, A)$ is measurable.
3. The Chapman-Kolmogorov equation is fulfilled for all $\tau_1,\ \tau_2 \geq 0$:

$$p_{\tau_1+\tau_2}(x, A) = \int_{\mathbb{X}} p_{\tau_1}(x, \mathrm{dy}) p_{\tau_2}(y, A).$$

4. For $\tau = 0$, the measure $p_0(x, \cdot)$ is the Dirac measure at $x$.

## Transition Kernel

### Definition

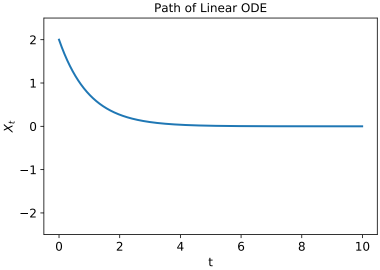
For a transition kernel $p_\tau$ and a probability measure $\nu_0$ on $\mathbb{X}$, define a Markov process via the finite-dimensional distributions

$$\begin{aligned}&\mathbb{P}(X_{t_l} \in A_l, \dots, X_{t_1} \in A_1)\\ &:= \int_{A_l} \dots \int_{A_1} \int_{\mathbb{X}} p_{t_l - t_{l-1}}(x_{l-1}, \mathrm{dx}_l) \dots p_{t_1}(x_0, \mathrm{dx}_1)\nu_0(\mathrm{dx}_0),\end{aligned}$$

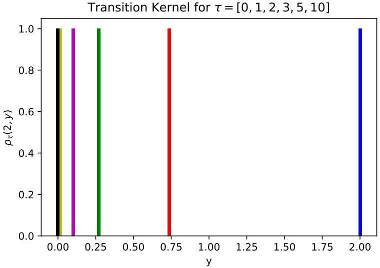
for time indices $0 \leq t_1 < \dots < t_l$ and $A_1, \dots A_l \subset \mathbb{X}$.

# Example 1

Deterministic Flow $X_t = \Phi^t(X_0)$ with transition kernel $p_\tau(x, \cdot) = \delta_{\Phi^t(X_0)}$.



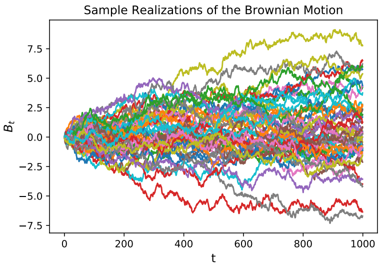

Linear ODE $\dot{X}_t = -X_t, X_0 = 2$.

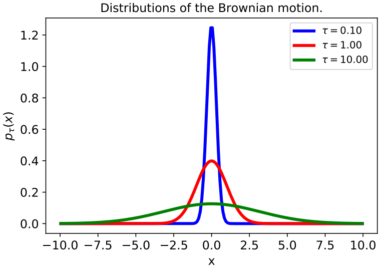



Distributions at times $\tau \in \{0, 1, 2, 3, 5, 10\}$.

# Example 2

Brownian motion with transition kernel $p_\tau(x, \cdot) \sim \mathcal{N}(x, \sqrt{\tau})$:




Paths of the Brownian motion started at $X_0 = 0$.

Distributions at times $\tau \in \{0.1, 1.0, 10.0\}$.

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Stationarity and Reversibility

# Perron-Frobenius Operator

## Definition

Let $\mu$ be a probability measure. The *Perron-Frobenius operator* is a linear operator $\mathcal{P}^\tau : L^1_\mu \mapsto L^1_\mu$, defined by

$$\int_A [\mathcal{P}^\tau f](x)\,\mathrm{d}\mu(x) = \int_{\mathbb{X}} f(x) p_\tau(x, A)\,\mathrm{d}\mu(x).$$

The operator $\mathcal{P}_t$ maps densities to densities. Hence, it describes the evolution of the probability distribution of the process associated to $p_\tau$.

## Koopman Operator

### Definition

Let $\mu$ be a probability measure. The *Koopman operator* is the adjoint of $\mathcal{P}^\tau$ defined on $L^\infty_\mu$ by:

$$\mathcal{K}^\tau f(x) = \int_{\mathbb{X}} p_\tau(x, \mathrm{d}y) f(y) = \mathbb{E}^x[f(X_\tau)].$$

Knowledge of the Koopman operator allows to make predictions about system statistics in the future.

## Examples

- Flow Map $X_t = \Phi^t(X_0)$:

$$\mathcal{K}^\tau f(x) = f(\Phi^t(x)).$$

- Brownian Motion:

$$\mathcal{K}^\tau f(x) = \frac{1}{\sqrt{2\pi\tau}} \int_{\mathbb{R}} f(y) \exp\left(-\frac{(y-x)^2}{2\tau}\right) \,\mathrm{d}y.$$

The Koopman operator allows the study of a complex system by means of **linear** operator.

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## Stationarity and Reversibility

# Invariant Measure

## Definition

Let $\mu$ be a probability measure. The measure $\mu$ is called **stationary** or **invariant** if for all $\tau \geq 0$ and $A \subset \mathbb{X}$:

$$\mu(A) = \int_{\mathbb{X}} p_\tau(x, A) \, \mathrm{d}\mu(x).$$

In other words, the distribution of the system does not change in time.

## Koopman on Hilbert Space

### Proposition

*Let $\mu$ be stationary for a transition kernel $p_\tau$. Then the Koopman operator can be extended to the Hilbert space $L^2_\mu(\mathbb{X})$. This extension is the adjoint of the Perron-Frobenius operator restricted to $L^2_\mu(\mathbb{X})$.*

### Proof.

For $f \in L^2_\mu$, Jensen's inequality shows that

$$\begin{aligned}||\mathcal{K}^\tau f||_2^2 &= \int_{\mathbb{X}} \left[\mathbb{E}^x[f(X_\tau)]\right]^2 \,\mathrm{d}\mu(x) \leq \int_{\mathbb{X}} \mathbb{E}^x\left[f(X_\tau)^2\right] \,\mathrm{d}\mu(x) \\ &= \int_{\mathbb{X}} \int_{\mathbb{X}} p_\tau(x, \mathrm{d}y)|f(y)|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) = \int_{\mathbb{X}} |f(y)|^2 \,\mathrm{d}\mu(y).\end{aligned}$$

# Reversibility

## Definition

Let $p_\tau$ be a stochastic transition kernel and $\mu$ be a measure. Then $p_\tau$ is said to be **reversible** with respect to $\mu$ if for all sets $A$, $B$:

$$\int_A p_\tau(x, B) \, \mathrm{d}\mu(x) = \int_B p_\tau(x, A) \, \mathrm{d}\mu(x),$$

which is the same as requiring $\mathbb{P}^\mu(X_0 \in A, X_\tau \in B) = \mathbb{P}^\mu(X_0 \in B, X_\tau \in A)$, i.e. there is no preferred direction in the system.

# Reversibility

For a reversible system, the Koopman operator becomes a self-adjoint operator on the Hilbert space $L^2_\mu(\mathbb{X})$, and $\mathcal{P}^\tau = \mathcal{K}^\tau$ for all $\tau \geq 0$.

# References

Lasota and Mackey, *Chaos, Fractals, and Noise* (2013)

Bauer, *Probability Theory* (2011)

Rogers and Williams, *Diffusions, Markov Processes, and Martingales* (2000)

## Solutions

(i)

$$\mathbb{P}(X_\tau \in A) = \int_{\mathbb{X}} p_\tau(x, A)\,\mathrm{d}\mu(x) = \int_A p_\tau(x, \mathbb{X})\,\mathrm{d}\mu(x) = \mu(A).$$

(ii) Set $f = \mathbf{1}_A, g = \mathbf{1}_B$ for measurable $A, B \subset \mathbb{X}$.

$$\begin{aligned}
\langle \mathcal{K}^\tau f,\, g\rangle_\mu &= \int_B \mathcal{K}^\tau f(x)\,\mathrm{d}\mu(x) = \int_B \int_{\mathbb{X}} f(y) p_\tau(x, \mathrm{d}y)\,\mathrm{d}\mu(x) \\
&= \int_B p_\tau(x, A)\,\mathrm{d}\mu(x) = \int_A p_\tau(x, B)\,\mathrm{d}\mu(x) \\
&= \langle f,\, \mathcal{K}^\tau g\rangle_\mu .
\end{aligned}$$