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Numerische Lineare Algebra 2 – Low Rank ADI Iteration.

Consider the Lyapunov equation

$$AX + XA^T = -BB^T, \quad \text{with} \quad A \in \mathbb{R}^{n\times n} \quad \text{large,sparse}, \quad \Lambda(A) \subset \mathbb{C}_-, \quad B \in \mathbb{R}^{n\times m},\ m \ll n. \tag{1}$$

The following algorithm [1, 2, 4] successively forms a low rank solution factor of the solution $X \approx ZZ^*$.

Algorithm 1 Low-rank ADI (LR-ADI) iteration for Lyapunov equations

Input: A, B forming (1), shifts $\mathcal{P} = \{p_1, \ldots, p_{\text{maxiter}}\} \subset \mathbb{C}_-$
Output: Z_j such that $X = Z_j Z_j^*$, $Z_j \in \mathbb{C}^{n\times mj}$ (approx.) solves (1).

1: $W_0 := B$, $j = 1$, $Z_0 = [\ ]$.
2: **while** $\|W_{j-1}\|_2^2 \geq$tol. **do**
3: Solve $(A + p_j I)V_j = W_{j-1}$ for $V_j \in \mathbb{C}^{n\times m}$.
4: $W_j = W_{j-1} - 2\,\mathrm{Re}(p_j)\,V_j$
5: $Z_j = \left[Z_{j-1},\ \sqrt{-2\,\mathrm{Re}(p_j)}V_j\right]$
6: $j = j + 1$
7: **end while**

Remarks:

- Major work are linear system solves with $(A+p_jI)$ and m right hand sides in each step in line 3. If linear solves (sparse-direct or iterative) are of $\approx$ linear complexity in n, complexity of LR-ADI is $\approx$ linear in n, linear in m.
- Requires complex arithmetic if the shifts are complex (can be reduced if $p_i \in \mathbb{C}_-$, $p_{i+1} = \overline{p_i}$ [3, 4]).
- For generalized Lyapunov equations $AXE^T + EXA^T = -BB^T$,$E \in \mathbb{R}^{n\times n}$ nonsingular, the lines 3 and 4 change to $V_j = (A + p_j E)^{-1} W_{j-1}$ and $W_j = W_{j-1} - 2\,\mathrm{Re}(p_j)\,EV_j$.
- Generalizations to Sylvester equations can be found in, e.g., [5, 6].

References

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