Otto-von-Guericke-Universität Magdeburg
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Numerical Linear Algebra – Question sheet 5.

Please hand in your completed problems on 24.01.2020.

Problem 1 **(7 points)**

For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{n \times n}$ unless otherwise indicated, "ew" stands for eigenvalue, "ev" stands for eigenvector.

- if λ is an ew of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an ew of $A - \mu I$
- if A is real and λ is an ew of A, then so is $-\lambda$
- if A is real and λ is an ew of A, then so is $\bar{\lambda}$
- if λ is an ew of A and A is nonsingular, then λ^{-1} is an ew of A^{-1}
- if all the ew's of A are zeros, then $A = 0$
- if A is Hermitian and λ is an ew of A, then $|\lambda|$ is a singular value of A
- if A is diagonalizable and all its ew's are equal, then A is diagonal

Problem 2 **(3 points)**

Let $A \in \mathbb{R}^{n \times n}$ be tridiagonal and symmetric, with all its sub- and superdiagonal entries nonzero. Prove that the eigenvalues of A are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $A - \lambda I$ has rank at least $n - 1$.)

Problem 3 **(4 points)**

Let $A \in \mathbb{C}^{n \times n}$ and let $\varepsilon > 0$, show that the following conditions are equivalent:

- z is an eigenvalue of $A + \delta A$ for some δA with $\|\delta A\|_2 \leq \varepsilon$
- there exists a vector $u \in \mathbb{C}^n$ with $\|Au - zu\|_2 \leq \varepsilon$
- the smallest singular value of $zI - A$, $\sigma_n(zI - A) \leq \varepsilon$
- $\|(zI - A)^{-1}\|_2 \geq \varepsilon^{-1}$

(if z is an eignevalue of A, we use the convention $\|(zI - A)^{-1}\|_2 = \infty$.)

Problem 4 **(2 points)**

What happens if you apply the unshifted QR algorithm to an orthogonal matrix?

Problem 4 **(4 points)**

Let $A \in \mathbb{R}^{n \times n}$ be symmetric, the *Rayleigh quotient* fucntion associated with A is defined as:

$$
\begin{aligned}
r \; &: \mathbb{R}^n \setminus 0 \to \mathbb{R} \\
x &\mapsto \frac{x^\top A x}{x^\top x}
\end{aligned}
$$

compute the gradient of r, $\nabla(r) = (\frac{\partial r}{\partial x_1}, \dots, \frac{\partial r}{\partial x_n})^\top$. What are the stationary points of r (i.e., the vectors x such that $\nabla(r)(x) = 0$, what are the corresponding values of r. We call the range of r the *field of values* of the matrix A and we denote it $W(A)$; show that $W(A)$ is equal to the convex hull of the eigenvalues of A.