Otto-von-Guericke-Universität Magdeburg
Max-Planck-Institut für Dynamik komplexer technischer Systeme
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Numerical Linear Algebra – Question sheet 4.

Please hand in your completed problems on 20.12.2019.

Problem 1 **(2 points)**

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definit (SPD) matrix. Let $v_1, \ldots, v_k$ be eigenvectors of A and let $b = \sum_{i=1}^{k} \alpha_k v_k$, where $\alpha_k \neq 0$. Show that the maximal dimension of the Krylov subspace generated by A and b is k?

Problem 2 **(5 points)**

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive semi-definit matrix. Let $b \in \text{Range}(A)$.
Consider the linear system of equations

$$Ax = b.$$

- show that this system has an infinite number of solutions, what is the set of solutions?
- prove that CG provided an initial guess $x_0 = 0$ converges to the solution that has minimal norm in the set of solutions.

Problem 3 **(13 points)**

Let $A, M \in \mathbb{R}^{n \times n}$ be SPD matrices. Let $Z \in \mathbb{R}^{n \times k}$ be a full rank matrix and let $E = Z^\top AZ$. Let x^* be the exact solution the linear system of equations

$$Ax = b.$$

We define the following operators:

$$P_B = (I - ZE^{-1}Z^\top A)M^{-1}(I - AZE^{-1}Z^\top) + ZE^{-1}Z^\top,$$
$$P_D = I - AZ(Z^\top AZ)^{-1}Z^\top,$$
$$P_{CM} = M^{-1} + ZE^{-1}Z^\top.$$

- show that P_B is SPD
- show that P_D is a projection
- show that $AP_D^\top = P_D A$
- compute $P_D AZ$
- show that CG, initiated with $x_0 = 0$, it finds an approximate solution of $P_D Ax = P_D b$
- show that $(I - P_D^\top)x_k = 0$
- express $(I - P_D^\top)x^*$ by using Z and E
- let $\hat{x}$ be the approximate solution given by CG for the linear system of equations $P_D Ax = P_D b$ with initial guess $x_0 = 0$. Give an expression of the associated approximate solution of the linear system of equations $Ax = b$.