Otto-von-Guericke-Universität Magdeburg
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Numerical Linear Algebra – Question sheet 3.

Please hand in your completed problems on 06.12.2019.
Send the code files to aldaas@mpi-magdeburg.mpg.de
(please use as a mail subject: NLA19 Homework III-FIRST NAME LAST NAME)

Problem 1 **(3 points)**

Let $M \in \mathbb{R}^{n\times n}$ be an SPD matrix. Let $\langle\cdot,\cdot\rangle_M$ be defined as:

$$\begin{aligned} f : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (x, y) &\mapsto \langle x, y\rangle_M = y^\top M x. \end{aligned}$$

- Prove that f defines an inner product on $\mathbb{R}^n$, we call f the M-inner product.
- Prove that

$$\begin{aligned} N : \mathbb{R}^n &\to \mathbb{R}_+ \\ x &\mapsto \sqrt{f(x,x)} = \sqrt{\langle x, x\rangle_M} = \sqrt{x^\top M x} \end{aligned}$$

defines a norm on $\mathbb{R}^n$.

Problem 2 **(2 points)**

Let $A, M \in \mathbb{R}^{n\times n}$ be SPD matrices. Show that the spectra of the matrices $M^{-1}A$, AM^{-1}, and $L^{-1}AL^{-\top}$ are the same where $M = LL^\top$.

Problem 3 **(4 points)**

Let $A, M \in \mathbb{R}^{n\times n}$ be SPD matrices and let $b \in \mathbb{R}^n$.

- Show that the matrix $M^{-1}A$ is SPD with respect to the M-inner product.
- Show that the matrix $M^{-1}A$ is SPD with respect to the A-inner product.
- With respect to which inner products is the matrix AM^{-1} SPD?

Now, we consider the following linear systems of equations

$$Ax = b, \tag{1}$$
$$L^{-1}AL^{-\top}w = L^{-1}b, \tag{2}$$
$$M^{-1}Ay = M^{-1}b, \tag{3}$$
$$AM^{-1}u = b, \tag{4}$$

where $M = LL^\top$.

- Give the relations between (a) y and x, (b) u and x, (c) w and x.

- For which linear systems of equations from the list above can we use the conjugate gradient (CG) algorithm with the Euclidean inner product? Explain why not when we cannot.

Problem 4 (6 points)

Write the CG algorithm that solves the linear system of equations (3). Use the following notation at iteration j, α_j is the step, p_j is the search direction, x_j is the approximate solution, z_j is the residual β_j orthogonality condition parameter. (Hint: replace the Euclidean inner product with the M-inner product.)
At iteration j of your algorithm, let $r_j = b - Ax_j$.

- What is the relation between r_j and z_j.
- Expand the M-inner products in the algorithm (i.e., rewrite $\langle v, q\rangle_M$ as $\langle Mv, q\rangle = q^\top Mv$) and reformulate the algorithm without showing any operation including the product of the matrix M. (Hint: use the auxiliary vector r_j.)

Define the auxiliary vectors and matrix:

$$\hat{p}_j = L^\top p_j,$$
$$\hat{u}_j = L^\top x_j,$$
$$\hat{r}_j = L^\top z_j,$$
$$\hat{A} = L^{-1}AL^{-\top}.$$

Rewrite the algorithm by using the previous notation and compare it to the CG algorithm for solving (2).

Problem 5 (5 points)

Let A_n be $n \times n$ matrix

$$A_n = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

and let $b_n = \frac{1}{\sqrt{n}}(1, \ldots, 1)^\top \in \mathbb{R}^n$. Write the algorithm you developed in the previous question to solve (3) where $A = A_{2n}$, $M = \begin{pmatrix} A_n & \\ & A_n \end{pmatrix}$ and $b = b_{2n}$. The interface of the algorithm should be as follows:

```
function [x, res, nIter] = myAwesomePCG(A, b, x_0, maxIter, tol, M)
```

where `x` is the returned approximate solution, `res` is the history of the residual norm, `nIter` is the number of iteration performed, `x_0` is the initial guess, `maxIter` is the maximal iteration count, and `tol` is the tolerance for the convergence of the algorithm based on the residual norm.
(Hint: to generate the matrix A_n you can use the function `toeplitz`; $A_n =$ `(sparse(toeplitz(t_n))` where $t_n = (2, -1, 0, \ldots, 0) \in \mathbb{R}^n$)
Generate the necessary results to fill in the following table (set `x_0` $= 0$, `maxIter` $= \frac{n}{10}$, `tol` $= 10^{-6}$).

n	`nIter`	$\|b - Ax\|_2$
100		
200		
400		
800		