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Fakultät für Mathematik – Institut für Analysis und Numerik
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Computational Methods for Systems and Control Theory

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Numerical Linear Algebra – 1. Hausaufgabe

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Note: unless otherwise stated, the norm of a matrix A, $\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$, where $\|\cdot\|$ is the Euclidean norm.

Exercise 1 (2 Punkte) (Finite and exact arithmetic)

As we know, addition in exact arithmetic is a commutative mathematical operation, i.e.,

$$a + b = b + a, \ \forall a, \ b \in \mathbb{R}.$$

Is the addition operation on floating point numbers commutative in general? If not, either give a contradictory example or describe how that cannot be commutative.

Exercise 2 (9 Punkte) (Absolute and relative condition numbers)

Let f be an operator from the vector space X equiped with the norm $\|\cdot\|_X$ to the vector space Y equiped with the norm $\|\cdot\|_Y$. We define the absolute condition number $\hat{\kappa}$ and the relative condition number κ of the operator f as

$$\hat{\kappa} = \lim_{\delta\to 0} \sup_{\|\delta x\|_X < \delta} \frac{\|f(x+\delta x) - f(x)\|_Y}{\|\delta x\|_X},$$
$$\kappa = \lim_{\delta\to 0} \sup_{\|\delta x\|_X < \delta} \left(\frac{\|f(x+\delta x) - f(x)\|_Y}{\|f(x)\|_Y}\right) / \left(\frac{\|\delta x\|_X}{\|x\|_X}\right).$$

1. Explain briefly what these quantities represent
2. compute the absolute and relative condition numbers corresponding to the function f in the following cases:
   - $f : \mathbb{R}^m \to \mathbb{R}^m$, $f(x) = 2x$
   - $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \sqrt{x}$
   - $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = Ax$, where $A \in \mathbb{R}^{m\times n}$, and $x \in \mathbb{R}^n$ (result depends on $\|A\|, \|x\|$, and $\|A\|$)
3. let $f : \Omega \subset\to \mathbb{R}^n$, $f(A) = x = A^{-1}b$, where $\Omega \subset \mathbb{R}^{n\times n}$ subset of invertible matrices, and $b \in \mathbb{R}^n$. We consider in this case a perturbation δA in the the matrix A. This perturbation introduces a perturbation δx in the output vector x, i.e., the following relation holds

   $$(A + \delta A)(x + \delta x) = b.$$

   - Prove that

     $$\delta x = -A^{-1}(\delta A x)$$

     and conclude that

     $$\|\delta x\| \leq \|A^{-1}\|\|A\|\|x\|.$$

- Prove that the relative condition number of f verifies the following inequality

$$\kappa \leq \|A^{-1}\|\|A\|.$$

Exercise 3 **(9 Punkte)** **(Sensitivity of eigenvector numerical computation)**

Let $\varepsilon > 0$ and let $A_\varepsilon = \begin{bmatrix} 1 & -\varepsilon^2 \\ -1 & 1 \end{bmatrix}$. Compute the characteristic polynomial of A_ε,

$$p_{A_\varepsilon}(x) = \det(xI - A_\varepsilon).$$

Compute the eigenvalues of the matrix A_ε and conclude that for $\varepsilon > 0$ the matrix is diagonalizable.
Compute the eigenvectors of A_ε and the spectral form of it.
Now let $\varepsilon = 0$ and compute the eigenvalues of A_0. How many eigenvectors does A_0 have? Does it admit a spectral form?
What can we say on the conditioning of the operator f that to each diagonalizable matrix associates the set of its eigenvectors?