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Computational Methods for Systems and Control Theory

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Model Reduction of Dynamical Systems - 5

Task: 1 (Solving algebraic Riccati equations via the matrix sign function)

Motivated by balancing-related methods such as LQG balanced truncation, let us consider the algebraic Riccati equation

$$A^T X + XA - XFX + G = 0,$$

where $A \in \mathbb{R}^{n\times n}$ and $F = F^T,\ G = G^T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices and (A, F) is stabilizable. Let $M = \begin{bmatrix} A^T & G \\ F & -A \end{bmatrix}$ and assume that the matrix sign function of M is partitioned as $\operatorname{sign}(M) = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$. Show that it holds $\begin{bmatrix} Z_{11} - I_n \\ Z_{21} \end{bmatrix} X = -\begin{bmatrix} Z_{12} \\ Z_{22} - I_n \end{bmatrix}$.

Hint: First show that it holds

$$M = \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \begin{bmatrix} (A-FX)^T & 0 \\ 0 & -(A-FX) \end{bmatrix} \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix}^{-1},$$

where Q solves $(A - FX)Q + Q(A - FX)^T + F = 0$. Then make use of the properties of the matrix sign function.

Task: 2 (The rational Krylov method for the Lyapunov equation)

Consider the Lyapunov equation

$$AX + XA^T + bb^T = 0,$$

with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Assume that a sequence of $m+1$ shift parameters $S = \{s_1, \dots, s_m, s_{m+1}\} \subset \mathbb{C}$ is given and that a projection matrix $V \in \mathbb{R}^{n\times m}$ is constructed according to the following procedure.

Algorithm 1 Rational Krylov Method by Ruhe

Require: A, b, S
Ensure: $V_{m+1} \in \mathbb{R}^{n\times m+1},\ T_m = V_m^* A V_m \in \mathbb{R}^{m\times m},\ H \in \mathbb{R}^{m+1\times m}$.
1: Compute $\tilde{v}_1 = (A - s_1 I)^{-1} b$.
2: Set $v_1 = \tilde{v}_1 / ||\tilde{v}_1||$.
3: **for** $j = 2 : m+1$ **do**
4: $\quad r = (A - s_j I)^{-1}\, v_{j-1}$.
5: $\quad h_{j-1} = V_{j-1}^* r$.
6: $\quad r = r - V_{j-1} h_{j-1}$
7: $\quad h_{j,j-1} = ||r||$.
8: $\quad v_j = r / h_{j,j-1}$.
9: **end for**
10: Set $T_m = V_m^* A V_m$.

Let $D_m = \operatorname{diag}(s_2, \dots, s_{m+1}) \in \mathbb{R}^{m\times m}$. Show that it holds

$$T_m = (I_m + H_m D_m - V_m^* A v_{m+1} h_{m+1,m} e_m^T) H_m^{-1}.$$

Here, e_m denotes the m-th unit vector and H_m consists of the first m columns of the matrix H.

Assume now that Y_m is a solution of the reduced Lyapunov equation

$$T_m Y_m + Y_m T_m^T + V_m^* b b^T V_m = 0.$$

Show that if $\tilde{v}_1 = b$, the residual $R_m = A V_m Y_m V_m^* + V_m Y_m V_m^* A^T + b b^T$ satisfies

$$||R_m||_F = ||S J S^T||_F, \ J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where S is the 3×3 upper triangular matrix in the QR factorization of

$$U = \begin{bmatrix} v_{m+1} s_{m+1} & V_m Y_m H_m^{-T} e_m h_{m+1,m} & -(I - V_m V_m^*) A v_{m+1} \end{bmatrix}.$$