Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
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Model Reduction for Dynamical Systems

— Lectures 5/6 —

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Outline

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Balanced Truncation

Basic principle:

- Recall: a stable system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

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- Compute balanced realization of the system via state-space transformation

$$\begin{aligned}\mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right)\end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

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Motivation:

The HSVs $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are system invariants: they are preserved under

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in transformed coordinates, the Gramians satisfy

$$\begin{aligned} (TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \end{aligned}$$

$$\Rightarrow \ (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda(PQ) = \Lambda((TPT^T)(T^{-T}QT^{-1}))$.

Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S, \; Q = R^T R.$
2. Compute SVD $SR^T = [\, U_1, \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

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Note:

$$V^T W \;\; = \;\; (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}})$$

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$$\begin{aligned} V^T W &= (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}} \\ &= \Sigma_1^{-\frac{1}{2}} [\, I_r, \, 0 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} \end{aligned}$$

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$\Longrightarrow VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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Balanced Truncation

Theoretical Background

Linear, Time-Invariant (LTI) Systems

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\begin{aligned}
\dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\
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Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

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State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

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- Recall: $A \in \mathbb{R}^{n\times m}$ is a linear operator, $A : \mathbb{R}^m \to \mathbb{R}^n$!
- Basic Idea: use SVD approximation as for matrix A!
- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

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Alternative to State-Space Operator: Hankel Operator

Instead of

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use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

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$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0.$

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$\leadsto$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0.$

$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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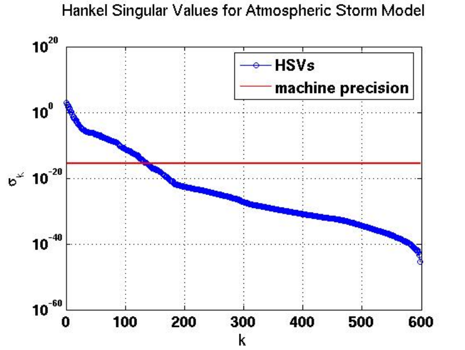

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$\Rightarrow$ Best approximation problem w.r.t. 2-induced operator norm well-posed

$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally unfeasible for large-scale systems.

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Theorem

Let P, Q be the controllability and observability Gramians of an LTI system Σ. Then the Hankel singular values $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the singular values of the Hankel operator associated to Σ.

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Singular values of $\mathcal{H}$ = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y_+(t) = \int_0^{\infty} B^T e^{A^T(\tau-t)} C^T y_+(\tau)\,d\tau$$

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$$y_+(t) = \mathcal{H}u_-(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu_-(\tau)\,d\tau = Ce^{At}z.$$

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$\Longleftrightarrow \quad PQz = \sigma^2 z.$ □

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Theorem

Let the reduced-order system $\hat{\Sigma} : (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ with $r \leq \hat{n}$ be computed by balanced truncation. Then the reduced-order model $\hat{\Sigma}$ is balanced, stable, minimal, and its HSVs are $\sigma_1, \ldots, \sigma_r$.

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Proof: Note that in balanced coordinates, the Gramians are diagonal and equal to

$$\mathrm{diag}(\Sigma_1, \Sigma_2) = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n).$$

Hence, the Gramian satisfies

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^T = 0,$$

whence we obtain the "controllability Lyapunov equation" of the reduced-order system,

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}^T + B_1 B_1^T = 0.$$

The result follows from $\hat{A} = A_{11}$, $\hat{B} = B_1$, $\Sigma_1 > 0$, the solution theory of Lyapunov equations and the analogous considerations for the observability Gramian. (Minimality is a simple consequence of $\hat{P} = \Sigma_1 = \hat{Q} > 0$.)

Singular Perturbation Approximation (aka Balanced Residualization)

Assume the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = [\, C_1, \; C_2 \,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

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Balanced truncation would set $x_2 = 0$ and use (A_{11}, B_1, C_1, D) as reduced-order model, thereby the information present in the remaining model is ignored!

Particularly, if $G(0) = \hat{G}(0)$ ("zero steady-state error") is required, one can apply the same condensation technique as in Guyan reduction: instead of $x_2 = 0$, set $\dot{x}_2 = 0$. This yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u, \\ y &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u, \end{aligned}$$

with

- the same properties as the reduced-order model w.r.t. stability, minimality, error bound, but $\hat{D} \neq D$;
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Note:

- A_{22} invertible as in balanced coordinates, $A_{22}\Sigma_2 + \Sigma_2 A_{22}^T + B_2 B_2^T = 0$ and (A_{22}, B_2) controllable, $\Sigma_2 > 0 \Rightarrow A_{22}$ stable.
- If the original system is not balanced, first compute a minimal realization by applying balanced truncation with $r = \hat{n}$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts '76, Moore '81]

- P = controllability Gramian of system given by (A, B, C, D).
- Q = observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman '83]

- $P/Q =$ controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

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Balanced Stochastic Truncation (BST) [Desai/Pal '84, Green '88]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

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Positive-Real Balanced Truncation (PRBT) [Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^TQ + Q\bar{A} + QB\bar{R}^{-1}B^TQ + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

Balancing-Related Methods

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT:} \quad \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT},$$

$$\text{LQGBT:} \quad \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST:} \quad \|G - G_r\|_\infty \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,$$

- Can be combined with singular perturbation approximation for steady-state performance.
- Computations can be modularized.