Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
Summer term 2015

Model Reduction for Dynamical Systems

— Lecture 4 —

Peter Benner Lihong Feng

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de feng@mpi-magdeburg.mpg.de

www.mpi-magdeburg.mpg.de/2909616/mor_ss15

Outline

Outline

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \operatorname{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Lemma 3.2 (Projector Properties)

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \dots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \dots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Lemma 3.2 (Projector Properties)

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \dots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Lemma 3.2 (Projector Properties)

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\,v_1, \ldots, v_r\,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Lemma 3.2 (Projector Properties)

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\,w_1, \ldots, w_r\,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\,v_1, \ldots, v_r\,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

Lemma 3.2 (Projector Properties)

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\; \forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\,w_1, \ldots, w_r\,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

Projection and Interpolation

Methods:

1. Modal Truncation
2. Balanced Truncation
3. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Model Reduction by Projection

Projection and Interpolation

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Model Reduction by Projection

Projection and Interpolation

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!
Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^T x$

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T \left(VW^T \dot{x} - AVW^T x - Bu\right)$$

Model Reduction by Projection

Projection and Interpolation

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^T x$

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$\begin{aligned} W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) &= W^T\left(VW^T\dot{x} - AVW^Tx - Bu\right) \\ &= \underbrace{W^T\dot{x}}_{\dot{\hat{x}}} - \underbrace{W^TAV}_{=\hat{A}}\underbrace{W^Tx}_{=\hat{x}} - \underbrace{W^TB}_{=\hat{B}}u \end{aligned}$$

Model Reduction by Projection

Projection and Interpolation

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!
Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^T x$

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$\begin{aligned} W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) &= W^T\left(VW^T\dot{x} - AVW^Tx - Bu\right) \\ &= \underbrace{W^T\dot{x}}_{\dot{\hat{x}}} - \underbrace{W^TAV}_{=\hat{A}}\underbrace{W^Tx}_{=\hat{x}} - \underbrace{W^TB}_{=\hat{B}}u \\ &= \dot{\hat{x}} - \hat{A}\hat{x} - \hat{B}u = 0. \end{aligned}$$

Model Reduction by Projection

Projection and Interpolation

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

Model Reduction by Projection

Projection and Interpolation

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1} B + D \right) - \left(\hat{C}(sI_r - \hat{A})^{-1} \hat{B} + \hat{D} \right) \\ &= C \left((sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1} W^T \right) B \end{aligned}$$

Model Reduction by Projection

Projection and Interpolation

Projection ⇝ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\left((sI_n - A)^{-1} - V(sI_r - \hat{A})^{-1}W^T\right)B \\ &= C\big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\big)(sI_n - A)^{-1}B. \end{aligned}$$

Model Reduction by Projection

Projection and Interpolation

Projection ⇝ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \Big(C(sI_n - A)^{-1}B + D\Big) - \Big(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\Big) \\ &= C\big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\big)(sI_n - A)^{-1}B. \end{aligned}$$

If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s_*)$ is a projector onto $\mathcal{V}$:

$\mathrm{range}\,(P(s_*)) \subset \mathrm{range}\,(V)$, all matrices have full rank $\Rightarrow$ "=",

$$P(s_*)^2 = V(s_* I_r - \hat{A})^{-1}W^T(s_* I_n - A)V(s_* I_r - \hat{A})^{-1}W^T(s_* I_n - A)$$

Model Reduction by Projection

Projection and Interpolation

Projection ⇝ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1} B + D \right) - \left(\hat{C}(sI_r - \hat{A})^{-1} \hat{B} + \hat{D} \right) \\ &= C \Big(I_n - \underbrace{V(sI_r - \hat{A})^{-1} W^T (sI_n - A)}_{=:P(s)} \Big) (sI_n - A)^{-1} B. \end{aligned}$$

If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s_*)$ is a projector onto $\mathcal{V}$:

$\mathrm{range}\,(P(s_*)) \subset \mathrm{range}\,(V)$, all matrices have full rank $\Rightarrow$ "=",

$$\begin{aligned} P(s_*)^2 &= V(s_* I_r - \hat{A})^{-1} W^T (s_* I_n - A) V (s_* I_r - \hat{A})^{-1} W^T (s_* I_n - A) \\ &= V(s_* I_r - \hat{A})^{-1} \underbrace{(s_* I_r - \hat{A})(s_* I_r - \hat{A})^{-1}}_{=I_r} W^T (s_* I_n - A) = P(s_*). \end{aligned}$$

Model Reduction by Projection
Projection and Interpolation

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\big)(sI_n - A)^{-1}B. \end{aligned}$$

If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s_*)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

hence

$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

Model Reduction by Projection

Projection and Interpolation

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \Big(C(sI_n - A)^{-1}B + D\Big) - \Big(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\Big) \\ &= C\big(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\big)(sI_n - A)^{-1}B. \end{aligned}$$

Analogously, $= C(sI_n - A)^{-1}\big(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\big)B.$

If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $Q(s)^H$ is a projector onto $\mathcal{W} \Longrightarrow$

if $(s_* I_n - A)^{-*} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0,$

hence

$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

Model Reduction by Projection

Projection and Interpolation

Theorem [Grimme '97, Villemagne/Skelton '87]

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \operatorname{range}(V)$, or
- $(s_* I_n - A)^{-*} C^T \in \operatorname{range}(W)$,

then the interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

Note: extension to Hermite interpolation conditions later!

Outline

1. Introduction
2. Mathematical Basics
3. Model Reduction by Projection
4. Modal Truncation
   - Basic Principle
   - Dominant Pole Algorithm

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H A V = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \; C_2 \,], \quad \hat{D} = D.$$

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W}^H = T^{-1}(1:r, :), \quad W = \tilde{W}(V^H \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^H AV = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \, C_2 \,], \quad \hat{D} = D.$$

Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

Modal Truncation

Basic method:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\,\hat{C},\ C_2\,], \quad \hat{D} = D.$$

Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \le \|C_2\|\|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\,\hat{C},\ C_2\,] \begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2, \end{aligned}$$

Modal Truncation

Basic method:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\,\hat{C},\; C_2\,], \quad \hat{D} = D.$$

Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \le \|C_2\|\|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$G(s) \quad = \quad \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \mathrm{diag}\left(\frac{1}{\jmath\omega - \lambda_{r+1}}, \ldots, \frac{1}{\jmath\omega - \lambda_n}\right) B_2.$$

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then reduced-order model is

$$\hat{A} := W^H AV = \mathrm{diag}\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^H B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\,\hat{C},\ C_2\,], \quad \hat{D} = D.$$

Difficulties:

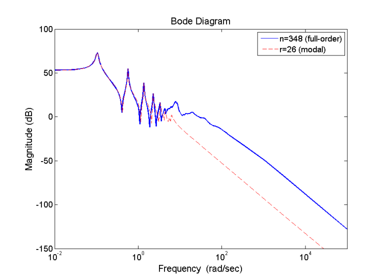
- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  ([Litz '79] use Jordan canoncial form; otherwise merely heuristic criteria, e.g., [Varga '95]. Recent improvement: dominant pole algorithm.)
- Error bound not computable for really large-scale problems.

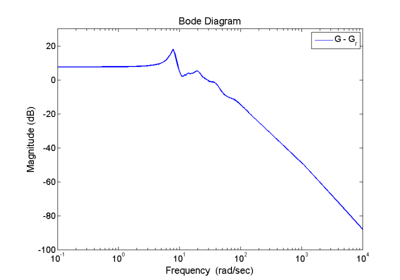
Basic Principle

Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = q = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function



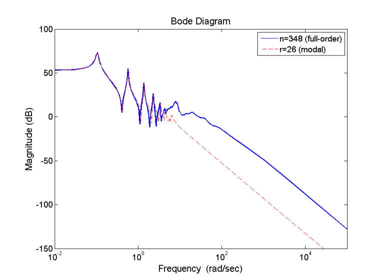


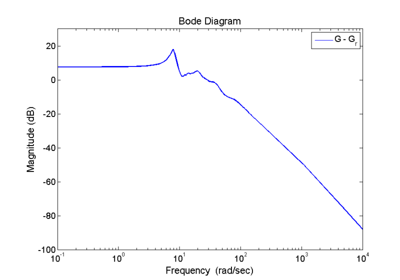
MATLAB® demo.

Basic Principle
Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = q = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \le 1.21 \cdot 10^3$

Bode plots of transfer functions and error function





MATLAB® demo.

Basic Principle
Extensions

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1}b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$.

Basic Principle

Extensions

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$\begin{aligned} 0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u. \end{aligned}$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\ y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u. \end{aligned}$$

Modal Truncation

Dominant Pole Algorithm

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

Modal Truncation

Dominant Pole Algorithm

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [\, x_1, \dots, x_n]$ the right and $X^{-1} =: Y = [y_1, \dots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned} G(s) &= C(sI - XDX^{-1})^{-1}B = CX(sI - \operatorname{diag}\{\lambda_1, \dots, \lambda_n\})^{-1} YB \\ &= [\, Cx_1, \dots, Cx_n \,] \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \\ &= \sum_{k=1}^{n} \frac{(Cx_k)(y_k^H B)}{s - \lambda_k}. \end{aligned}$$

Modal Truncation

Dominant Pole Algorithm

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

Note: $R_k = (Cx_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\mathrm{res}\,(G, \lambda_\ell) = \lim_{s \to \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^{n} \underbrace{\lim_{s \to \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{= \begin{cases} 0 \text{ for } k \neq \ell \\ 1 \text{ for } k = \ell \end{cases}} R_k = R_\ell.$$

Modal Truncation

Dominant Pole Algorithm

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

As projection basis use spaces spanned by right/left eigenvectors corresponding to dominant poles, i.e.. $(\lambda_j,\ x_j,\ y_j)$ with largest

$$\|R_k\| / |\operatorname{re}(\lambda_k)|.$$

Modal Truncation

Dominant Pole Algorithm

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{q \times m}$.

As projection basis use spaces spanned by right/left eigenvectors corresponding to dominant poles, i.e.. $(\lambda_j,\ x_j,\ y_j)$ with largest

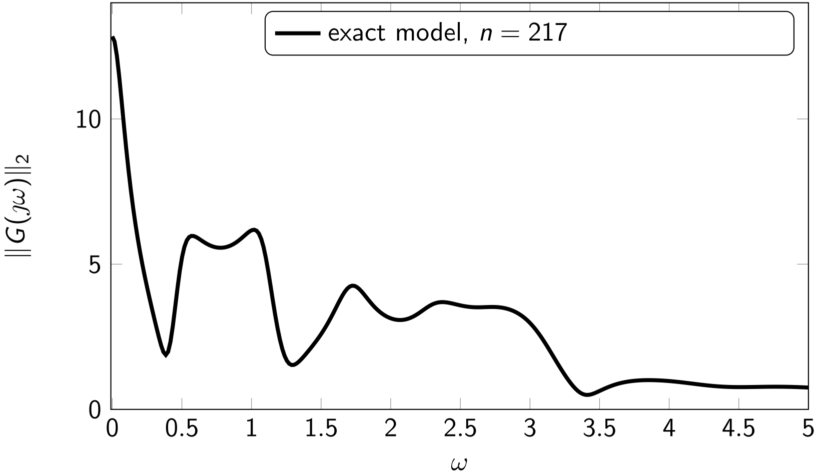
$$\|R_k\| / |\operatorname{re}(\lambda_k)|.$$

Remark

The dominant modes have most important influence on the input-output behavior of the system and are responsible for the "peaks"' in the frequency response.

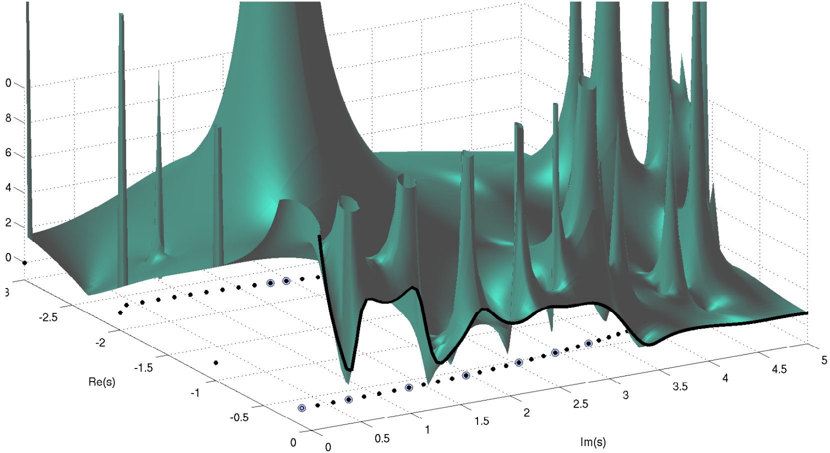
Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



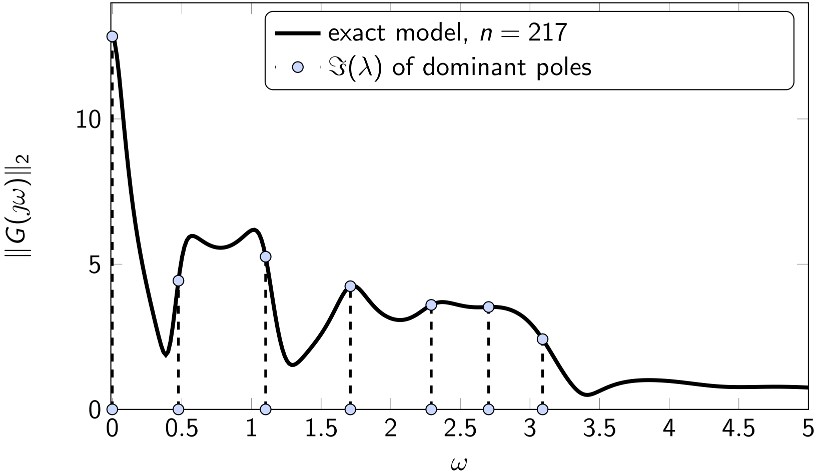
Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



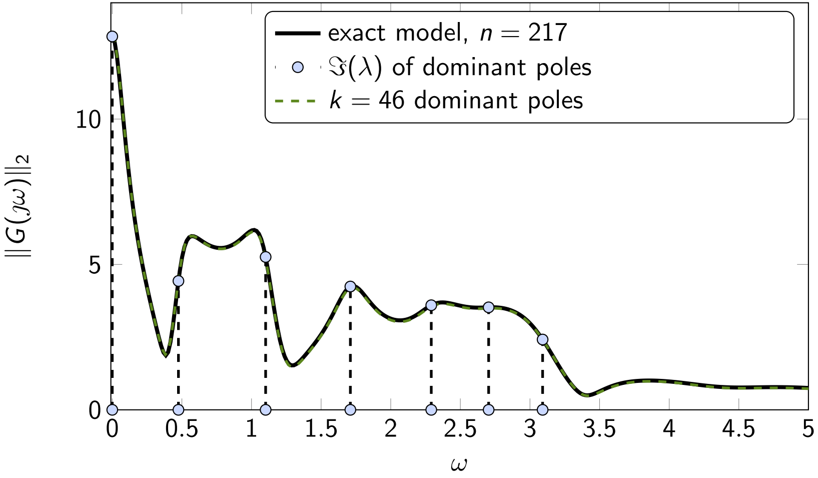
Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$**)**



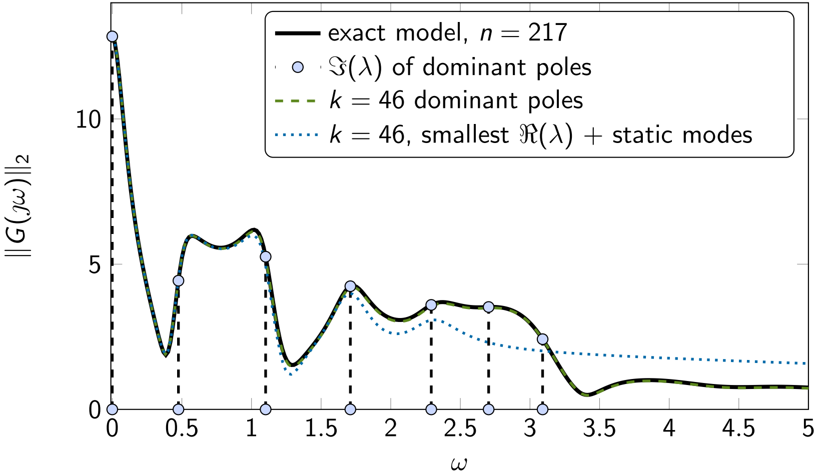
Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



Dominant Poles

Random SISO Example ($B,\ C^T \in \mathbb{R}^n$)



Dominant Poles

Random SISO Example ($B, \ C^T \in \mathbb{R}^n$)

Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominante Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.