Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
Summer term 2015

Model Reduction for Dynamical Systems

— Lectures 2/3 —

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Outline

Numerical Linear Algebra

Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory (in single precision): $4 \cdot n_x \cdot n_y$ bytes.

Theorem (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Numerical Linear Algebra

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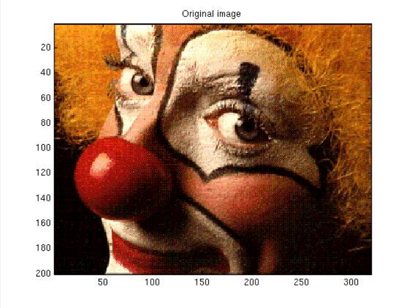
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel
$\rightsquigarrow \quad \approx 256$ kB

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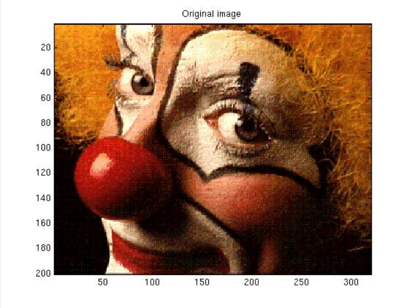

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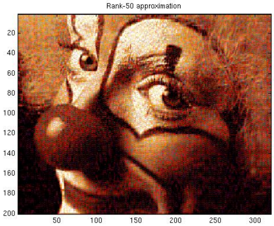
- rank $r = 50$, ≈ 104 kB



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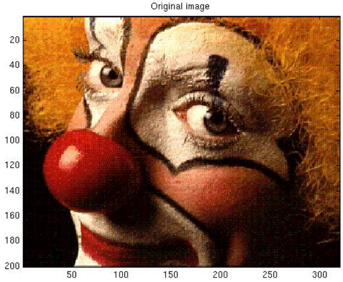

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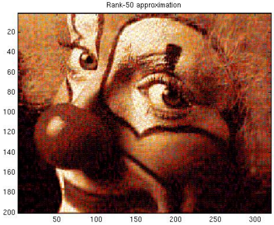
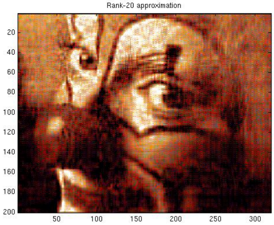


- rank $r = 20$, ≈ 42 kB

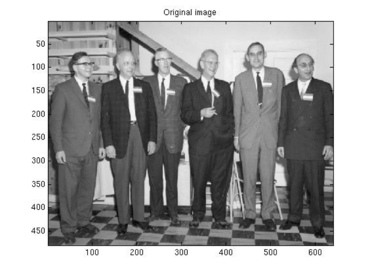


Dimension Reduction via SVD

Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens,
George Forsythe, Alston Householder,
Peter Henrici, Fritz L. Bauer.



640 $\times$ 480 pixel, $\approx$ 1229 kB

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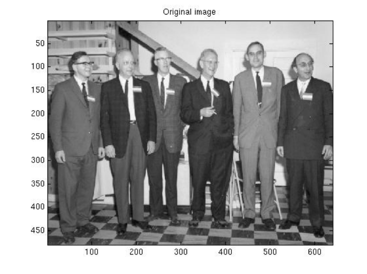
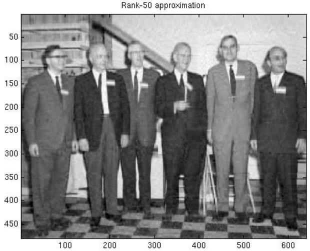


640×480 pixel, ≈ 1229 kB

- rank $r = 100$, ≈ 448 kB

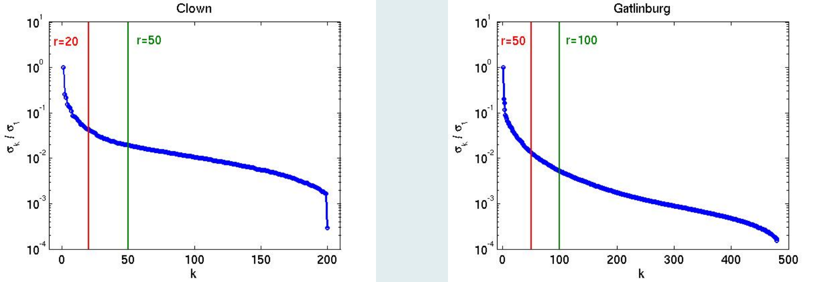
- rank $r = 50$, ≈ 224 kB



Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



A different viewpoint

Linear Mapping

A matrix $A \in \mathbb{R}^{\ell \times k}$ represents a linear mapping

$$\mathcal{A} : \mathbb{R}^k \to \mathbb{R}^\ell \; : \; x \to y := Ax.$$

The truncated SVD ignores small Hankel singular values and thus the related left and right singular vectors.

Consequence:

- Vectors (almost) in the kernel of A do not contribute to $\mathrm{range}\,(A)$ and can hardly or not at all be reconstructed from the input-output relation ("A^{-1}") $\rightsquigarrow$ "unobservable" states.
- Vectors (almost) in $\mathrm{range}\,(A)^\perp$ cannot be "reached" from any $x \in \mathbb{R}^k$ $\rightsquigarrow$ "unreachable/uncontrollable" states.
- Hence, the truncated SVD ignores states hard to reconstruct and hard to reach.

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- Hence, the truncated SVD ignores states hard to reconstruct and hard to reach.

Systems and Control Theory
The Laplace transform

Definition

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f \mapsto F, \quad F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

F is a function in the (Laplace or) frequency domain.

Note: for frequency domain evaluations ("frequency response analysis"), one takes $\mathrm{re}\, s = 0$ and $\mathrm{im}\, s \geq 0$. Then $\omega := \mathrm{im}\, s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi v$ with v measured in [Hz]).

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Lemma

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s) - f(0).$$

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Note: for ease of notation, in the following we will use lower-case letters for both, a function and its Laplace transform!

Systems and Control Theory
The Model Reduction Problem as Approximation Problem in Frequency Domain

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s))$ to linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sEx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

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$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sE - A)^{-1}B + D}_{=:G(s)} \Big) u(s).$$

$G(s)$ is the transfer function of Σ.

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Goal: Fast evaluation of mapping $u \rightarrow y$.

Systems and Control Theory

The Model Reduction Problem as Approximation Problem in Frequency Domain

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} E\dot{x} &= Ax + Bu, \qquad E, A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \hat{E}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{E}, \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

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of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G}) \leq r} \|G - \hat{G}\|.$

Systems and Control Theory
Properties of linear systems

Definition

A linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of $A - \lambda E$, denoted by $\Lambda(A, E)$, satisfies $\Lambda(A, E) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

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Systems and Control Theory
Properties of linear systems

Further properties to be discussed:

- Controllability/reachability
- Observability
- Stabilizability
- Detectability

Systems and Control Theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad \text{with transfer function } G(s) = C(sI - A)^{-1}B + D,$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x \rightarrow Tx, \\ (A, B, C, D) \rightarrow (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

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Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_2 \in \mathbb{R}^{q \times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

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Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

Systems and Control Theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

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Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Systems and Control Theory

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Theorem

A realization (A, B, C, D) of a linear system is minimal $\iff$
(A, B) is controllable and (A, C) is observable.

Systems and Control Theory
Balanced Realizations

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

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When does a balanced realization exist?

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U\Sigma V^T$ is the SVD of SR^T.

Proof. Exercise!

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Systems and Control Theory
Balanced Realizations

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

Systems and Control Theory
Balanced Realizations

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The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

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Proof. In balanced coordinates, the HSVs are $\Lambda\,(PQ)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

This is equivalent to

$$0 = A(T^{-1}\hat{P}T^{-T}) + (T^{-1}\hat{P}T^{-T})A^T + BB^T.$$

The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda\,(\hat{P}\hat{Q}) = \Lambda\,(PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

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Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the L_2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\,d\omega.$$

Assume A (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \mathrm{re}\,z < 0\}$.

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Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \leq M < \infty \Rightarrow$

$$\int_{-\infty}^{\infty} y(\jmath\omega)^H y(\jmath\omega)\,d\omega \quad = \quad \int_{-\infty}^{\infty} u(\jmath\omega)^H G(\jmath\omega)^H G(\jmath\omega) u(\jmath\omega)\,d\omega$$

(Here, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

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&= \quad \int_{-\infty}^{\infty} \|G(\jmath\omega)u(\jmath\omega)\|^2\, d\omega \leq \int_{-\infty}^{\infty} M^2 \|u(\jmath\omega)\|^2\, d\omega \\
&= \quad M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega \;<\; \infty.
\end{aligned}$$

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\end{aligned}$$

$$\Longrightarrow y \in \mathcal{L}_2^q \cong L_2^q(-\infty, \infty).$$

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Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\,(G(\jmath\omega))\,.$$

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Sketch of proof:
$\|G(\jmath\omega)u(\jmath\omega)\| \leq \|G(\jmath\omega)\|\|u(\jmath\omega)\| \Rightarrow$ "$\leq$".
Construct u with $\|Gu\|_2 = \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\|\|u\|_2$.

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Hardy space $\mathcal{H}_\infty$

Function space of matrix-/scalar-valued functions that are analytic and bounded in $\mathbb{C}^+$.
The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\mathrm{re}\, s > 0} \sigma_{\max}\left(F(s)\right) = \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = q = 1$);
- $\mathcal{H}_\infty^{q\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, q > 1$).

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\left(sI - A\right)^{-1} B + D.$$

Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}u\|_2 \leq \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider stable transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned} \|F\|_2 \quad &:= \quad \frac{1}{2\pi}\left(\sup_{\operatorname{re}\sigma>0}\int_{-\infty}^{\infty}\|F(\sigma+\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}} \\ &= \quad \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\|F(\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

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Qualitative and Quantitative Study of the Approximation Error

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$\mathcal{H}_2$ approximation error for impulse response ($u(t) = u_0\delta(t)$)

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \leq \|G - \hat{G}\|_2\|u_0\|.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_2 < tol$!

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$$\|F\|_2 \quad = \quad \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2\, d\omega\right)^{\frac{1}{2}}.$$

Theorem (Practical Computation of the $\mathcal{H}_2$-norm)

$$\|F\|_2^2 = \operatorname{tr}\left(B^T Q B\right) = \operatorname{tr}\left(C P C^T\right),$$

where P, Q are the controllability and observability Gramians of the corresponding LTI system.

Qualitative and Quantitative Study of the Approximation Error
Approximation Problems

Output errors in time-domain

$$\begin{aligned} \|y - \hat{y}\|_2 &\leq \|G - \hat{G}\|_\infty \|u\|_2 \qquad \Longrightarrow \|G - \hat{G}\|_\infty < \mathrm{tol} \\ \|y - \hat{y}\|_\infty &\leq \|G - \hat{G}\|_2 \|u\|_2 \qquad \Longrightarrow \|G - \hat{G}\|_2 < \mathrm{tol} \end{aligned}$$

Qualitative and Quantitative Study of the Approximation Error

Approximation Problems

Output errors in time-domain

$$\|y - \hat{y}\|_2 \leq \|G - \hat{G}\|_\infty \|u\|_2 \implies \|G - \hat{G}\|_\infty < \mathrm{tol}$$

$$\|y - \hat{y}\|_\infty \leq \|G - \hat{G}\|_2 \|u\|_2 \implies \|G - \hat{G}\|_2 < \mathrm{tol}$$

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Qualitative and Quantitative Study of the Approximation Error
Computable error measures

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}(\text{value})$.

  For MIMO systems, $q \times m$ array of plots G_{ij}.