Summary of moment-matching methods

Consider a linear time invariant (LTI) system

$$\begin{cases} E\dot{x} = Ax + Bu(t), \\ y = Cx, \end{cases} \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_I}$, $C \in \mathbb{R}^{p\times n_O}$. The transfer function is $H(s) = C(sE - A)^{-1}B$. The matrix E could be a singular matrix, i.e. not invertible.

1 Explicit moment-matching method: AWE

For single-input single-output (SISO) system:

$H(s) - \hat{H}(s) = O(s^{2q})$.

$\hat{H}(s) = \frac{P_{q-1}(s)}{Q_q(s)}$

Moments matched: $m_i = \hat{m}_i$, $i = 0, 1, \ldots, 2q - 1$.

AWE method approximates the transfer function $H(s)$ by a rational function $\hat{H}(s) = \frac{P_{q-1}(s)}{Q_q(s)}$, which is the Padé approximation of $H(s)$.

The method explicitly computes the moments of the transfer function, which is not numerically stable. Usually, the accuracy of the approximation cannot be improved for $q \geq 8$.

2 Implicit moment-matching method

2.1 Padé approximation

$$\begin{aligned} \text{range}(V) &= \text{span}\{\tilde{B}(s_0), \tilde{A}(s_0)\tilde{B}(s_0), \ldots, (\tilde{A}(s_0))^{q-1}\tilde{B}(s_0)\}, \\ \text{range}(W) &= \text{span}\{C^T, \tilde{A}^T(s_0)C^T, \ldots, (\tilde{A}^T(s_0))^{q-1}C^T\}, \end{aligned} \tag{2}$$

where $\tilde{B}(s_0) = (s_0E - A)^{-1}B$, $\tilde{A}(s_0) = (s_0E - A)^{-1}$, and s_0 is the expansion point. We require $W^TV = I$, i.e. the matrices W and V are bi-orthogonal with each other.

2.1.1 Reduced model 1

The reduced model is constructed as

$$\begin{cases} W^T(s_0E - A)^{-1}EV\frac{dz}{dt} = W^T(s_0E - A)^{-1}AVz + W^T(s_0E - A)^{-1}Bu(t), \\ y = CVz. \end{cases} \tag{3}$$

Define $\tilde{E} = W^T(s_0E - A)^{-1}EV$, $\tilde{A} = W^T(s_0E - A)^{-1}AV$, $\tilde{B} = W^T(s_0E - A)^{-1}B$, and $\tilde{C} = CV$. Here, $\tilde{H}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}$ is the transfer function of the reduced model in (3).

- Moments matched: $m_i = \tilde{m}_i$, $i = 0, 1, \ldots, 2q - 1$ if the original system is a SISO system.
- If the original system in (1) is a SISO system, the transfer function is a Padé approximation of the original system, i.e. $H(s) - \tilde{H}(s) = O(s^{2q})$ or $m_i = \tilde{m}_i$, $i = 0, 1, \ldots, 2q - 1$.

2.1.2 Reduced model 2

The reduced model is constructed as

$$\begin{cases} W^T E V \frac{dz}{dt} = W^T A V z + W^T B u(t), \\ y = CVz. \end{cases} \tag{4}$$

Define $\hat{E} = W^T EV$, $\hat{A} = W^T AV$, $\tilde{B} = W^T B$, and $\tilde{C} = CV$. Here, $\hat{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ is the transfer function of the reduced model in (3).

- If the original system in (1) is a SISO system, and if $E = I$, the identity matrix, then the transfer function is a Padé approximation of the original system, i.e. $H(s) - \hat{H}(s) = O(s^{2q})$, or $m_i = \hat{m}_i$, $i = 0, 1, \ldots, 2q - 1$.
- Moments matched: $m_i = \hat{m}_i$, $i = 0, 1, \ldots, 2q - 1$, if $E = I$, and if the original system is a SISO system;

  $M_i = \hat{M}_i$, $i = 0, 1, \ldots, 2q - 1$ if $E = I$, and if the original system is a MIMO system.

2.2 Matrix-Padé approximation

- For the reduced model in (3), assume that $z \in R^r$, i.e. the order of the reduced model is r, then $M_i = \tilde{M}_i, i = 0, 1, \ldots \left\lfloor \frac{r}{n_O} \right\rfloor + \left\lfloor \frac{r}{n_I} \right\rfloor$. Here n_O is the number of outputs, and n_I is the number of inputs.

2.3 Padé type approximation

The reduced model is constructed by Galerkin projection rather than Petrov-Galerkin projection as below:

$$\begin{cases} V^T E V \frac{dz}{dt} = V^T A V z + V^T B u(t), \\ y = CVz. \end{cases} \tag{5}$$

Define $\hat{E} = V^T EV$, $\hat{A} = V^T AV$, $\tilde{B} = V^T B$, and $\tilde{C} = CV$. V is defined as above. Here, $\hat{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ is the transfer function of the reduced model in (3).

- For both SISO and MIMO system in (1), the transfer function $\hat{H}(s)$ matches q moments of the original transfer function, i.e. $m_i = \hat{m}_i$, or $M_i = \hat{M}_i$, $i = 0, 1, \ldots, q - 1$.

2.4 Rational interpolation

Let

$$\begin{aligned} \text{range}(V) &= \text{span}\{\tilde{B}(s_0), \tilde{A}_B(s_0)\tilde{B}(s_0), \ldots, (\tilde{A}_B(s_0))^{q-1}\tilde{B}(s_0)\}, \\ \text{range}(W) &= \text{span}\{\tilde{C}^T, \tilde{A}_C(s_0)\tilde{C}^T, \ldots, (\tilde{A}_C(s_0))^{q-1}\tilde{C}^T\}, \end{aligned} \tag{6}$$

where $\tilde{B}(s_0) = (s_0 E - A)^{-1}B$, $\tilde{A}_B(s_0) = (s_0 E - A)^{-1}E$, $\tilde{A}_C(s_0) = (s_0 E - A)^{-T}E^T$, $\tilde{C}^T = (s_0 E - A)^{-T}C^T$.

The reduced model is constructed similarly as in (3), but with the matrices W and V in (6) instead.

- It is not required that $W^T V = I$ for rational interpolation.
- For both SISO and MIMO system in (1), the transfer function $\hat{H}(s)$ matches $2q$ moments of the original transfer function, i.e. $m_i = \tilde{m}_i$, or $M_i = \hat{M}_i$, $i = 0, 1, \ldots, 2q - 1$. In this sense, $\hat{H}(s)$ is also a Padé approximation of $H(s)$ for SISO system in (1).
- IRKA is a special case of rational interpolation, where $q = 1$ for W and V.