Parametric Model Order Reduction

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Outline

Parametric Systems

A linear parametric system

$$\begin{aligned} C(s_1, s_2, \cdots, s_{p-1})\frac{dx}{dt} &= G(s_1, s_2, \cdots, s_{p-1})x + Bu(t), \\ y(t) &= L^{\mathrm{T}}x, \end{aligned} \tag{1}$$

where the system matrices $C(s_1, s_2, \cdots, s_{p-1})$, $G(s_1, s_2, \cdots, s_{p-1})$ are (maybe, nonlinear, non-affine) functions of the parameters $s_1, s_2, \cdots, s_{p-1}$.

PMOR based on multi-moment matching (Krylov subspace PMOR)

In frequency domain

Using Laplace transform, the system in (1) is transformed into

$$\begin{aligned} E(s_1, \ldots, s_p)x &= Bu(s_p), \\ y &= L^{\mathrm{T}}x, \end{aligned} \tag{2}$$

where the matrix $E \in \mathbb{R}^{n\times n}$ is parametrized. The new parameter s_p is in fact the frequency parameter s, which corresponds to time t.

In case of a nonlinear and/or non-affine dependence of the matrix E on the parameters, the system in (2) is first transformed to an affine form

$$\begin{aligned} (E_0 + \tilde{s}_1 E_1 + \tilde{s}_2 E_2 + \ldots + \tilde{s}_p E_p)x &= Bu(s_p), \\ y &= L^{\mathrm{T}}x. \end{aligned} \tag{3}$$

Here the newly defined parameters $\tilde{s}_i, i = 1, \ldots, p$, might be some functions (rational, polynomial) of the original parameters s_i in (2).

PMOR based on multi-moment matching

To obtain the projection matrix V for the reduced model, the state x in (3) is expanded into a Taylor series at an expansion point $\tilde{s}_0 = (\tilde{s}_1^0, \ldots, \tilde{s}_p^0)^T$ as below,

$$\begin{aligned} x &= [I - (\sigma_1 M_1 + \ldots + \sigma_p M_p)]^{-1} \tilde{E}^{-1} B u(s_p) \\ &= \sum_{m=0}^{\infty} [\sigma_1 M_1 + \ldots + \sigma_p M_p]^m \tilde{E}^{-1} B u(s_p) \\ &= \sum_{m=0}^{\infty} \sum_{k_2=0}^{m-(k_3+\ldots+k_p)} \cdots \sum_{k_{p-1}=0}^{m-k_p} \\ &\quad \sum_{k_p=0}^{m} [F^m_{k_2,\ldots,k_p}(M_1, \ldots, M_p) B_M u(s_p) \sigma_1^{m-(k_2+\ldots+k_p)} \sigma_2^{k_2} \cdots \sigma_p^{k_p}, \end{aligned} \tag{4}$$

where $\sigma_i = \tilde{s}_i - \tilde{s}_i^0$, $\tilde{E} = E_0 + \tilde{s}_1^0 E_1 + \ldots + \tilde{s}_p^0 E_p$, $M_i = -\tilde{E}^{-1} E_i$, $i = 1, 2, \ldots p$, and $B_M = \tilde{E}^{-1} B$. The $F^m_{k_2,\ldots,k_p}(M_1, \ldots, M_p)$ can be generated recursively as

PMOR based on multi-moment matching

$$
\begin{aligned}
&F^m_{k_2,\ldots,k_p}(M_1,\ldots,M_p)\\
&=\begin{cases}
0, & \text{if } k_i \notin \{0,1,\ldots,m\}, i=2,\ldots,p,\\
0, & \text{if } k_2+\ldots+k_p \notin \{0,1,\ldots,m\},\\
I, & \text{if } m=0,\\
M_1F^{m-1}_{k_2,\ldots,k_p}(M_1,\ldots,M_p)+M_2F^{m-1}_{k_2-1,\ldots,k_p}(M_1,\ldots,M_p)+\ldots & \\
\qquad \ldots+M_pF^{m-1}_{k_2,\ldots,k_p-1}(M_1,\ldots,M_p), & \text{else.}
\end{cases}
\end{aligned}
$$

For example, if there are two parameters $\tilde{s}_1, \tilde{s}_2$ in (3), $F^m_{k_2,\ldots,k_p}(M_1,\ldots,M_p) = F^m_{k_2}$ are:

$$
\begin{aligned}
&F^0_0 = I,\\
&F^1_0 = M_1F^0_0 = M_1,\;\; F^1_1 = M_2F^0_0 = M_2 \qquad (5)\\
&F^2_0 = M_1F^1_0 = M_1^2,\;\; F^2_1 = M_1F^1_1 + M_2F^1_0 = M_1M_2 + M_2M_1,\;\; F^2_2 = M_2F^1_1 = M_2^2,\\
&\ldots
\end{aligned}
$$

PMOR based on multi-moment-matching

For the general case, the projection matrix V is constructed as

$$
\begin{aligned}
&\mathrm{range}\{V\} \\
&= \mathrm{colspan}\{\bigcup_{m=0}^{m_q} \bigcup_{k_2=0}^{m-(k_p+\ldots+k_3)} \cdots \bigcup_{k_{p-1}=0}^{m-k_p} \bigcup_{k_p=0}^{m} F^m_{k_2,\ldots,k_p}(M_1,\ldots,M_p)B_M\} \\
&= \mathrm{colspan}\{B_M, M_1B_M, M_2B_M, \ldots, M_pB_M, (M_1)^2B_M, (M_1M_2+M_2M_1)B_M, \ldots, \\
&\qquad (M_1M_p+M_pM_1)B_M, (M_2)^2B_M, (M_2M_3+M_3M_2)B_M, \ldots\}.
\end{aligned}
\tag{6}
$$

We call the coefficients in the series expansion of the state x in (4) the *moment vectors* of the parametric system. The corresponding moments of the transfer function are the moment vectors multiplied by L^{T} from the left. For example,

PMOR based on multi-moment-matching

- $L^{\mathrm{T}} B_M$: the 0th order multi-moments; the columns in B_M: the 0th order moment vectors.
- $L^{\mathrm{T}} M_i B_M,\ i = 1, 2, \dots, p$: the first order multi-moments; the columns in $M_i B_M,\ i = 1, 2, \dots, p$: the first order moment vectors.
- $\dots$; the columns in $M_i^2 B_M,\ i = 1, 2, \dots, p$, $(M_1 M_i + M_i M_1) B_M,\ i = 2, \dots, p$, $(M_2 M_i + M_i M_2) B_M,\ i = 3, \dots, p$, $\dots$, $(M_{p-1} M_p + M_p M_{p-1}) B_M$: the second order moment vectors.
- $\dots$.

Since the coefficients corresponding not only to $s = s_p$, but also to those associated with the other parameters $s_i, i = 1, \dots, p-1$ are, we call them as **multi-moments** of the transfer function.

A Robust Algorithm

Taking a closer look at the power series expansion of x in (4), we get the following equivalent, but different formulation,

$$
\begin{aligned}
x &= [I - (\sigma_1 M_1 + \ldots + \sigma_p M_p)]^{-1} \tilde{E}^{-1} B u \\
&= \sum_{m=0}^{\infty} [\sigma_1 M_1 + \ldots + \sigma_p M_p]^m B_M u \\
&= B_M u + [\sigma_1 M_1 + \ldots + \sigma_p M_p] B_M u + [\sigma_1 M_1 + \ldots + \sigma_p M_p]^2 B_M u + \ldots \\
&\quad + [\sigma_1 M_1 + \ldots + \sigma_p M_p]^j B_M u + \ldots
\end{aligned}
\tag{7}
$$

By defining

$$
\begin{aligned}
x_0 &= B_M, \\
x_1 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] B_M, \\
x_2 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p]^2 B_M, \ldots, \\
x_j &= [\sigma_1 M_1 + \ldots + \sigma_p M_p]^j B_M, \ldots,
\end{aligned}
$$

we have $x = (x_0 + x_1 + x_2 + \cdots + x_j + \cdots)u$ and obtain the recursive relations

A Robust Algorithm

$$
\begin{aligned}
x_0 &= B_M, \\
x_1 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_0, \\
x_2 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_1, \ldots \\
x_j &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_{j-1}, \ldots.
\end{aligned}
$$

If we define a vector sequence based on the coefficient matrices of $x_j,\ j = 0, 1, \ldots$ as below,

$$
\begin{aligned}
R_0 &= B_M, \\
R_1 &= [M_1 R_0, M_2 R_0, \ldots, M_p R_0], \\
R_2 &= [M_1 R_1, M_2 R_1, \ldots, M_p R_1], \\
&\vdots \\
R_j &= [M_1 R_{j-1}, M_2 R_{j-1}, \ldots, M_p R_{j-1}], \\
&\vdots
\end{aligned} \tag{8}
$$

A Robust Algorithm

and let R be the subspace spanned by the vectors in $R_j,\ j = 0, 1, \cdots, m$:

$$R = \mathrm{colspan}\{R_0, \ldots, R_j, \ldots, R_m\},$$

then there exists $z \in \mathbb{R}^q$, such that $x \approx Vz$. Here the columns in $V \in \mathbb{R}^{n \times q}$ is a basis of R. We see that the terms in $R_j,\ j = 0, 1, \ldots, m$ are the coefficients of the parameters in the series expansion (7). They are also the j-th order moment vectors.

How to compute an orthonormal basis V?

Algorithm 1 Compute $V = [v_1, v_2, \ldots, v_q]$

```
Initialize a_1 = 0, a_2 = 0, sum = 0.
Compute R_0 = \tilde{E}^{-1} B.
if (multiple input) then
    Orthogonalize the columns in R_0 using MGS: [v_1, v_2, ..., v_{q_1}] =
    orth{R_0} with respect to a user given tolerance ε > 0 specifying
    the deflation criterion for numerically linearly dependent vectors.
    sum = q_1            % q_1 is the number of columns remaining after
    deflation w.r.t. ε.)
else
    Compute the first column in V: v_1 = R_0/||R_0||_2
    sum = 1
end if
% Compute the orthonormal columns in R_1, R_2, ..., R_m iteratively as
below
```

Algorithm 2 Continued

for $i = 1, 2, \ldots, m$ **do**
  $a_2 = sum$;
  for $t = 1, 2, \ldots, p$ **do**
    IF $a_1 = a_2$, *stop* **ELSE do**
    for $j = a_1 + 1, \ldots a_2$ **do**
      $w = \tilde{E}^{-1} E_t v_j$; $col = sum + 1$;
      for $k = 1, 2, \ldots, col - 1$ **do**
        $h = v_k^T w$; $w = w - h v_k$
      end for
      if $\|w\|_2 > \varepsilon$ **then**
        $v_{col} = \frac{w}{\|w\|_2}$; $sum = col$;
      end if
    end for
  end for
  $a_1 = a_2$;
end for
Orthogonalize the columns in V by MGS w.r.t. ε.

Conclusions

- PMOR methods include also reduced basis methods, which is a huge topic. The curse of dimensionality of the parameters is still unsolved.
- Other PMOR methods are not introduced: POD based method, reduced basis method, tansfer function interpolation based method, PMOR based on measured data.

References

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