Outline

- Overlook
- Controllability measures
- Observability measures
- Infinite Gramians
- MOR: Balanced truncation based on infinite Gramians

Overlook

Balanced truncation: first balancing, then truncate.

Given a LTI system:
$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

For convenience of discussion, we denote the system as a block form:

$$\left(\begin{array}{c|c} A & B \\ \hline L^T & \end{array}\right) \xrightarrow{\text{Balancing } T} \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{L}^T & \end{array}\right) = \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{L}_1^T & \tilde{L}_2^T & \end{array}\right)$$

reduced model $\leftarrow \left(\begin{array}{c|c} \tilde{A}_{11} & \tilde{B}_1 \\ \hline \tilde{L}_1^T & \end{array}\right) \leftarrow$ truncate

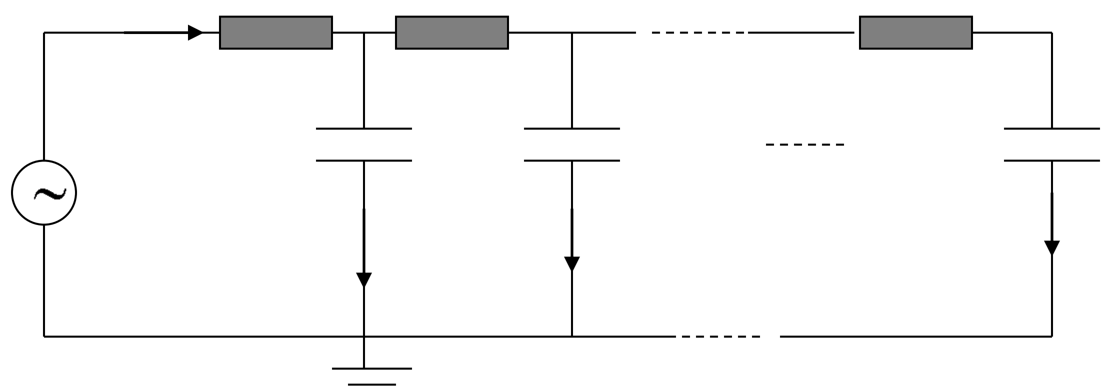
The unimportant part is truncated

Overlook

What’s the
unimportant part?

The states which are difficult to control and difficult to observe correspond the unimportant part.

In system theory, the unknown vector *x* is called the state of the system. Actually, the entries in *x* depict the system variables, such as branch currents, node voltages in the interconnect model, and therefore describe the state of the system.



Balanced Truncation

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Analytical solution of the LTI System

When discuss balanced truncation method, we limit the LTI system to the following form:

$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

In order to analyze controllability, observability, we need to use the analytical solution of the system, though we always solve the system numerically (i.e. by numerical methods and using computers).

The analytical solution of the system: the analytical representation of $x(t)$.

Analytical solution of the LTI System

What is the analytical solution excited by the input $u(t)$ and starting with the initial state $x(0) = x_0$?
(see also Chapter 4, section 4.2 in [Chi-Tsong Chen, Linear System Theory and Design, 3rd edition, 1999])

Multiplying e^{-At} on both sides of $dx(t)/dt = Ax(t) + Bu(t)$ yields

$$e^{-At}\frac{dx(t)}{dt} - e^{-At}Ax(t) = e^{-At}Bu(t)$$

which implies,

$$\frac{d}{dt}(e^{-At}x(t)) = e^{-At}Bu(t)$$

Its integration from 0 to t yields,

$$e^{-A\tau}x(\tau)\Big|_{\tau=0}^{t} = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

Analytical solution of the LTI System

Thus we have
$$e^{-At}x(t) - e^{0}x_0 = \int_0^t e^{-A\tau}Bu(\tau)d\tau \qquad (1)$$

Because the inverse of e^{-At} is e^{At} and $e^{0} = I$, (1) implies
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad (2)$$

This is the analytical solution of $dx(t)/dt = Ax(t) + Bu(t)$.

- It is impossible to plot the waveform of $x(t)$ by hand, we need computers to compute $x(t)$ numerically and plot $x(t)$ at many samples of time.
- It is difficult to compute $x(t)$ by following the analytical formulation in (2) if A is very large. We need to solve the LTI system numerically with some numerical methods, like backward Euler, ...etc.

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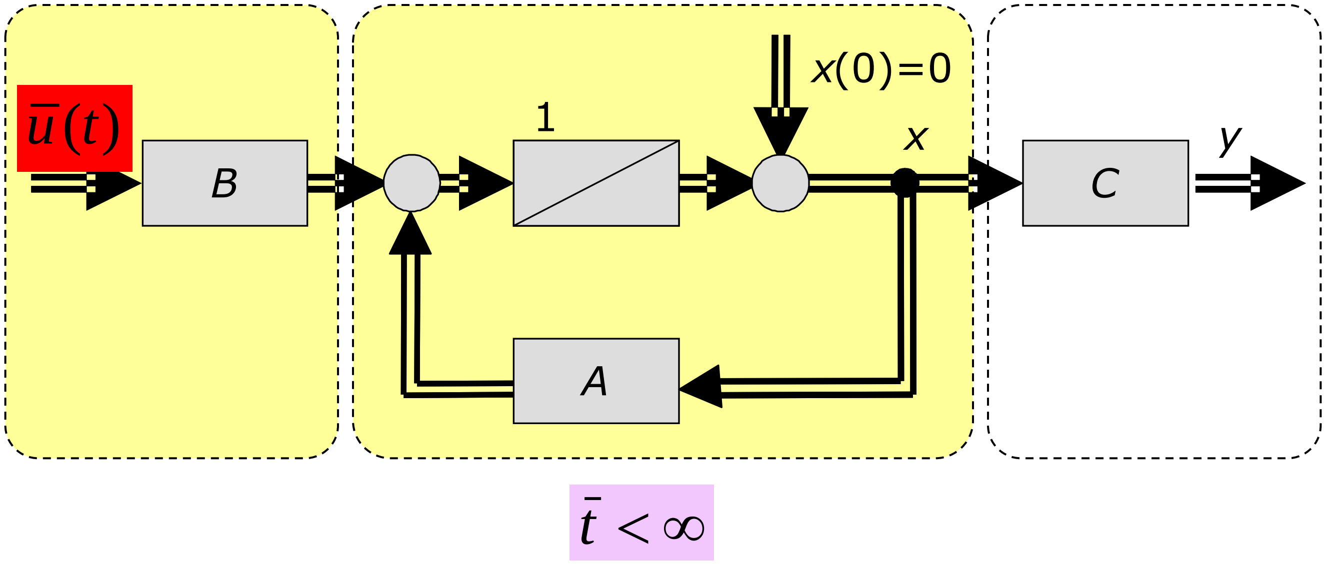
Controllability measure

Reachability

Definition: Given a system $\left(\begin{array}{c|c} A & B \\ \hline L^T & \end{array}\right)$, a state x is reachable from the zero state if there exist an input function $\bar{u}(t)$ of finite energy such that x can be obtain from the zero state and within a finite period of time $\bar{t} < \infty$.



Controllability measure

Denote X^{reach} the subspace spanned by the reachable states, then

$$X^{reach} \subseteq X$$

X is the whole state space, e.g.

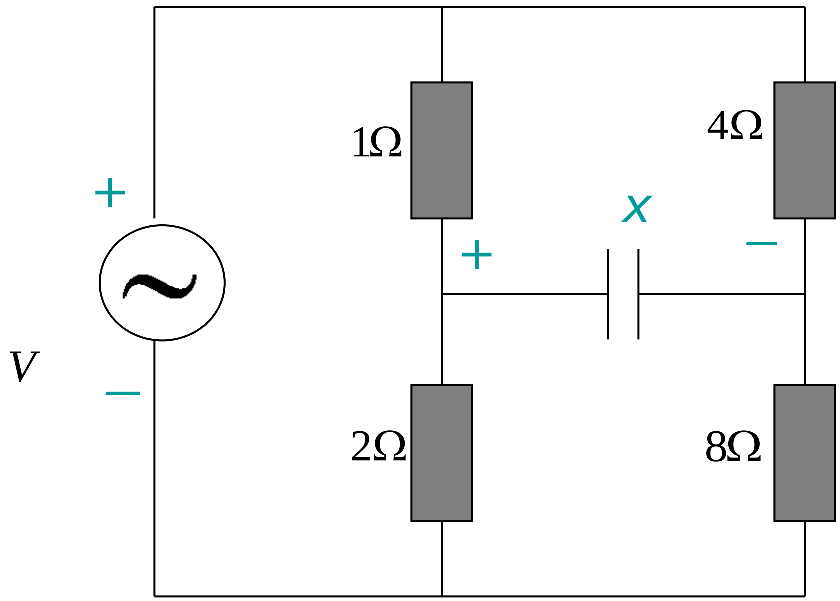
$$X = \{x(t) : R_+ \to C^n\}$$

The system is reachable $\Longleftrightarrow$ $X^{reach} = X$: every state in the state space is reachable.

Controllability measure

Example 1

Picture referred to [Chi-Tsong Chen, Linear system Theory and Design, 3rd edition, New York Oxford, Oxford University Press, 1999]



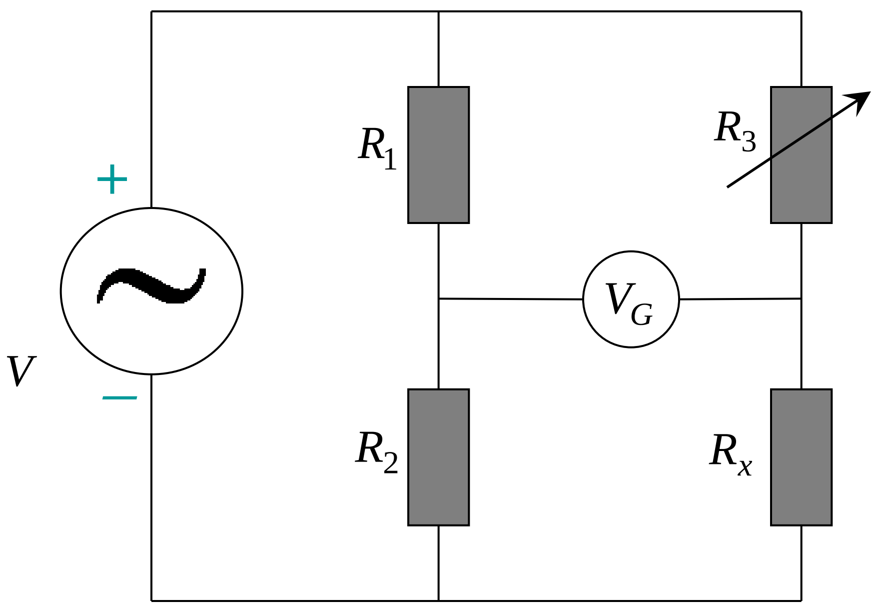
x denotes the voltage drop along the capacitor, and is the state of the system. In this circuit, x=0 at any time.

Conclusion:

In this circuit, 0 state is a reachable state, but any nonzero state is a unreachable state! Therefore the whole system is unreachable.

Controllability measure

Example 1 is actually the Wheatstone bridge.



Wheatstone bridge

R_3 is adjustable, it is adjusted till V_G becomes zero. It means there is no voltage drop through V_G.

Therefore, we have

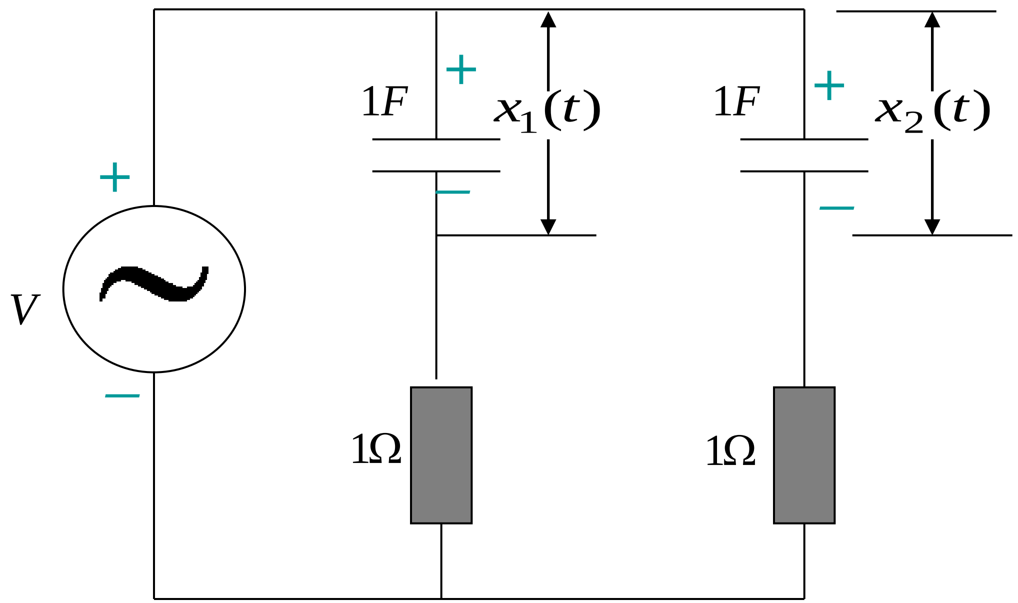
$$\frac{R_2}{R_1} = \frac{R_x}{R_3}$$

R_x can be easily measured by the above equation.

A **Wheatstone bridge** is a measuring instrument invented by Samuel Hunter Christie in 1833 and improved and popularized by Sir Charles Wheatstone in 1843. (http://en.wikipedia.org/wiki/Wheatstone_bridge)

Controllability measure

Example 2



$$dx(t)/dt = Ax(t) + Bu(t)$$

$$y(t) = L^T x(t)$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

voltage drops through the two capacitors.

Those states $x(t)$ with $x_1(t) = x_2(t)$ are reachable, but those states with $x_1(t) \neq x_2(t)$ are not reachable. Because whatever the input is, the voltage drops through the two capacitors are always identical.

Therefore the whole system is unreachable.

Controllability measure

Reachability matrix of the system:

$$R(A,B) = [B, AB, A^2B \cdots A^{n-1}B \cdots]$$

By the Cayley-Halmilton theorem, the rank of the reachability matrix and the span of its columns are determined (at most) by the first *n* terms (not the first *n* columns), i.e. $A^t B, t = 1,2,\cdots,n-1$.

Thus for computational purpose the following (finite) reachability matrix is of importance:

$$R_n(A,B) = [B, AB, A^2B \cdots A^{n-1}B]$$

Sometimes $R_n(A,B)$ is directly defined as the reachability matrix.

- Why it is called reachability matrix?
- Any connection between $R_n(A,B)$ and reachability?

Controllability measure

Notice the analytical solution of system state equation $dx/dt = Ax + Bu$ is

$$x(u, x_0, t) = e^{At} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau, t \geq t_0,$$

The reachability of a state *x* of the system is tested by the zero initial state, $x_0 = 0$, we look at the above analytical solution with $x_0 = 0$,

$$x(u, 0, t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

Notice:

$$e^{At} = I_n + \frac{t}{1!} A + \frac{t^2}{2!} A^2 + \cdots + \frac{t^k}{k!} A^k + \cdots$$

Controllability measure

$$x(u,0,t)=\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\int_0^t (B+(t-\tau)AB+\frac{(t-\tau)^2}{2!}A^2B+\ldots)u(\tau)d\tau$$

$$=B\int_0^t u(\tau)d\tau+AB\int_0^t (t-\tau)u(\tau)d\tau+A^2B\int_0^t \frac{(t-\tau)^2}{2!}u(\tau)d\tau$$

$$=B\alpha_0+AB\alpha_1+A^2B\alpha_2+\cdots+A^kB\alpha_k+\cdots,$$

which means a reachable state *x* is the linear combination of the terms:

$$B, AB, A^2B,\cdots, A^kB,\cdots$$

Therefore $R(A,B)=(B,A,A^2B\cdots A^{n-1}B\cdots)$ is defined as the reachability Matrix.

Controllability measure

Actually there is a Theorem (Theorem 4.5 in Chapter 4 in [Antoulas05]):

Theorem 1 If X^{reach} is the subspace spanned by the reachable states, then

$$X^{reach} = \text{im } R(A,B) : \text{space spanned by the columns.}$$

The theorem tells us the subspace spanned by all reachable states is exactly the subspace spanned by the columns of the reachability matrix $R(A,B)$.

The finite reachability gramian at time $t < \infty$ is defined as :

$$P(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau, \quad for \quad 0 < t < \infty$$

Controllability measure

Connection between reachability matrix and reachability gramians

Proposition 1 The finite reachability gramians have the following properties: (a) $P(t) = P^T(t) \geq 0,$ and (b) their columns span the reachability subspace, i.e., $\operatorname{im} P(t) = \operatorname{im} R(A, B).$ (Proposition 4.8 in [Antulous 05])

Proof An easier way is to prove $\operatorname{im} P^{\oplus}(t) = \operatorname{im} R^{\oplus}(A, B),$ where

$$\operatorname{im} P^{\oplus}(t) \oplus \operatorname{im} P(t) = C^n \quad \text{and} \quad \operatorname{im} R^{\oplus}(A, B) \oplus \operatorname{im} R(A, B) = C^n$$

We first prove $\forall x \in \operatorname{im} P^{\oplus}(t) \Rightarrow x \in \operatorname{im} R^{\oplus}(A, B)$

$\forall x \in \operatorname{im} P^{\oplus}$ we have

$$x^T P(t) x = \int_0^t \| B^T e^{A^T \tau} x \|^2 \, d\tau = 0,$$

$$\Leftrightarrow B^T e^{A^T t} x = 0, \; for\, all \quad t \geq 0$$

Controllability measure

$$e^{A^T t} = I_n + \frac{t}{1!}A^T + \frac{t^2}{2!}(A^T)^2 + \cdots + \frac{t^k}{k!}(A^T)^k + \cdots$$

Therefore, $B^T e^{A^T t} x = 0 \Leftrightarrow \boxed{B^T (A^T)^{i-1} x = 0,}$ for all $i > 0$.

$$\Downarrow$$

$$x \perp A^{i-1} B$$

$$\Downarrow$$

$$x \perp \operatorname{im} R(A, B)$$

$$\Updownarrow$$

$$x \in \operatorname{im} R^{\oplus}(A, B)$$

We have proved: $\forall x \in \operatorname{im} P^{\oplus}(t) \Rightarrow x \in \operatorname{im} R^{\oplus}(A, B)$

Controllability measure

Next we prove: $\forall x \in \operatorname{im} R^{\oplus}(A,B) \Rightarrow x \in \operatorname{im} P^{\oplus}$

$$x \in \operatorname{im} R^{\oplus}(A,B) \Longrightarrow x \perp \operatorname{im} R(A,B) \Longrightarrow x \perp A^{i-1}B, \textit{ for all } i > 0$$

$$\Downarrow$$

$$B^T (A^T)^{i-1} x = 0, \textit{ for all } i > 0.$$

$$\Downarrow$$

$$B^T e^{A^T t} x = 0, \textit{ for all } t \geq 0$$

$$\Downarrow$$

$$e^{At} B B^T e^{A^T t} x = 0$$

$$\Downarrow$$

$$P(t)x = \int_0^t e^{A\tau} B B^T e^{A^T \tau} x \, d\tau = 0,$$

$$\Longleftarrow$$

$$x \in null(P)$$

$$\Updownarrow \quad P \text{ symmetric}$$

$$x \perp \operatorname{im}(P)$$

why?

$$\Updownarrow$$

$$x \in \operatorname{im} P^{\oplus}$$

Controllability measure

$$\forall x \in \mathrm{null}(P) \Longleftrightarrow Px = 0 \Longleftrightarrow \begin{pmatrix} p_1^T x \\ p_2^T x \\ \vdots \\ p_n^T x \end{pmatrix} = 0 \; and \quad P = \begin{pmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_n^T \end{pmatrix}$$

$$\Longleftrightarrow p_i \perp \mathrm{null}(P) \Longleftrightarrow \mathrm{im}(P^T) \perp \mathrm{null}(P)$$

$$\mathrm{im}(P^T) = \mathrm{span}\{\text{columns of } P^T\} = \mathrm{span}\{p_1, \ldots, p_n\}$$

$\Downarrow$ *P* symmetric

$$\mathrm{im}(P) = \mathrm{im}(P^T)$$

$$\Downarrow$$

$$\mathrm{im}(P) \perp \mathrm{null}(P)$$

Controllability measure

The relation $\operatorname{im} P(t) = \operatorname{im} R(A, B)$ provides a way to derive the minimal energy which are needed to reach a state *x.*

The states using large minimal energy are difficult to reach and will be truncated during MOR based on balanced truncation.

Therefore, the minimal energy for reaching a reachable state *x* is a key concept for model order reduction based on balanced truncation.

Next, we will derive the minimal energy for reaching a state *x.*

Controllability measure

From the analytical solution, if a state x is reached at time $\bar{T}$, then $\exists u(t)$ with finite energy, such that

$$x = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B u(\tau) d\tau$$

How much must the input $u(t)$ be?

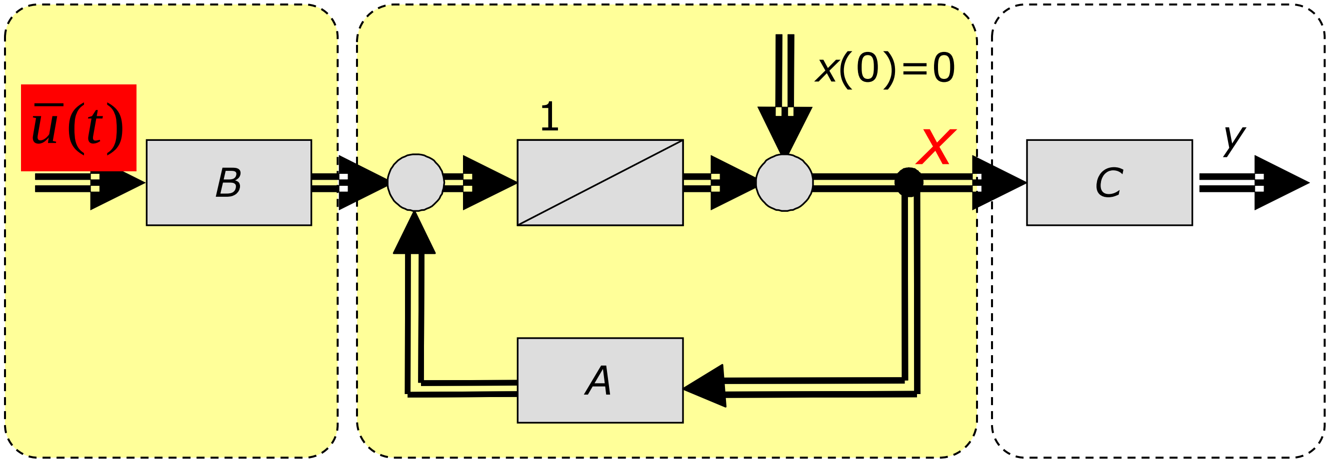
We have proved if x is reachable, then $x \in \operatorname{im}(P(t))$, i.e. $\exists \xi, \bar{T}$,

$$x = P(\bar{T})\xi \Rightarrow x = \int_0^{\bar{T}} e^{At} B B^T e^{A^T t} \xi dt = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B B^T e^{A^T(\bar{T}-\tau)} \xi d(-\tau)$$

$$= \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B \bar{u} d\tau \qquad and \qquad \bar{u}(\tau) = -B^T e^{A^T(\bar{T}-\tau)} \xi$$

This means x can be reached at time $\bar{T}$ with input $\bar{u}$

Controllability measure



$\bar{t} < \infty$

The input *u*(t) is the excitation of the system, its energy is the energy required to reach the state *x* .

Energy of a function is defined as: $\| u \|^2 = \int_0^{\bar{T}} u^*(t) u(t) dt$

Controllability measure

We see from above analysis, if x is reachable at time $\bar{t}$, *x* can be represented as:

$$x = \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B\bar{u}\, d\tau \qquad (\bar{u} = -B^T e^{A^T(\bar{t}-\tau)}\xi)$$

Any other input $\| u(t) \|^2 > \| \bar{u}(t) \|^2$ can also reach *x*. However if $\| u(t) \|^2 < \| \bar{u}(t) \|^2$, it cannot reach *x* at time $\bar{t}$, may need longer time.

Actually the energy of $\bar{u}$ is the minimal energy to reach the state *x* at the given time period $\bar{t}$. (Proposition 4.10 in [Antulous 05])

Energy of $\bar{u}$:

$$\| \bar{u} \|^2 = \int_0^{\bar{t}} \bar{u}^*(t)\bar{u}(t)dt = \int_0^{\bar{t}} \xi^* e^{A(\bar{t}-t)} BB^T e^{A^T(\bar{t}-t)} \xi dt = \xi^* P(\bar{t})\xi$$

relation to *x*?

↕

x

Controllability measure

A system is reachable means every state x in the whole state space is reachable.

From theorem 1: $X^{reach} = \text{im}\, R(A,B) = \text{im}\, R_n(A,B)$

Therefore the system is reachable $\Longleftrightarrow$ $\text{rank}\,(R_n(A,B)) = n$

From Proposition 1: $\text{im}\, P(t) = \text{im}\, R(A,B)$

Therefore the system is reachable $\Longleftrightarrow$ $\text{rank}\,(P(t)) = n, \forall t > 0$

Therefore, $P(t)$ is nonsingular for any *t*, if the system is reachable.

Controllability measure

Energy of $\bar{u} = B^T e^{A^T(\bar{t}-\tau)}\xi$ (notice $x = P(\bar{t})\xi$) :

$$\| \bar{u} \|^2 = \xi^* P(\bar{t})\xi = (P^{-1}(\bar{t})x)^* P(\bar{t})(P^{-1}(\bar{t})x) = x^* P^{-1}(\bar{t})x$$

$$\| \bar{u} \|^2 = x^* P^{-1}(\bar{t})x$$

Controllability measure!

Only for reachable systems.

Controllability measure

Remark 1:

Reachability is a generic property for LTI systems with the form:

$$dx/dt = Ax + Bu$$

This means, intuitively, that almost every LTI system with the form above is reachable. If there are any unreachable systems, they are very rare. The unreachable LTI systems like examples 1,2 are rare.

Remark 2:

The reachability of the system can be more easily checked by the criteria:

The system is reachable $\Longleftrightarrow$ $rank(R_n(A, B)) = n$

Controllability measure

A concept which is closely related to reachability is that of controllability.

Here, instead of driving the zero state to a desired state, a given non-zero state is steered to the zero state. More precisely we have:

Definition of controllability: Given a LTI system as above, a non-zero state x is controllable if there exist an input $u(t)$ with finite energy such that the state of the system goes to zero from x within a finite time: $\bar{t} < \infty$.

Controllability measure

It has been proved that for time continuous LTI systems (as discussed in this lecture), the concepts of reachability and controllability are equivalent.

Theorem 2 For time continuous systems $X^{reach} = X^{contr}$. (Theorem 4.16 in Antulous 05)

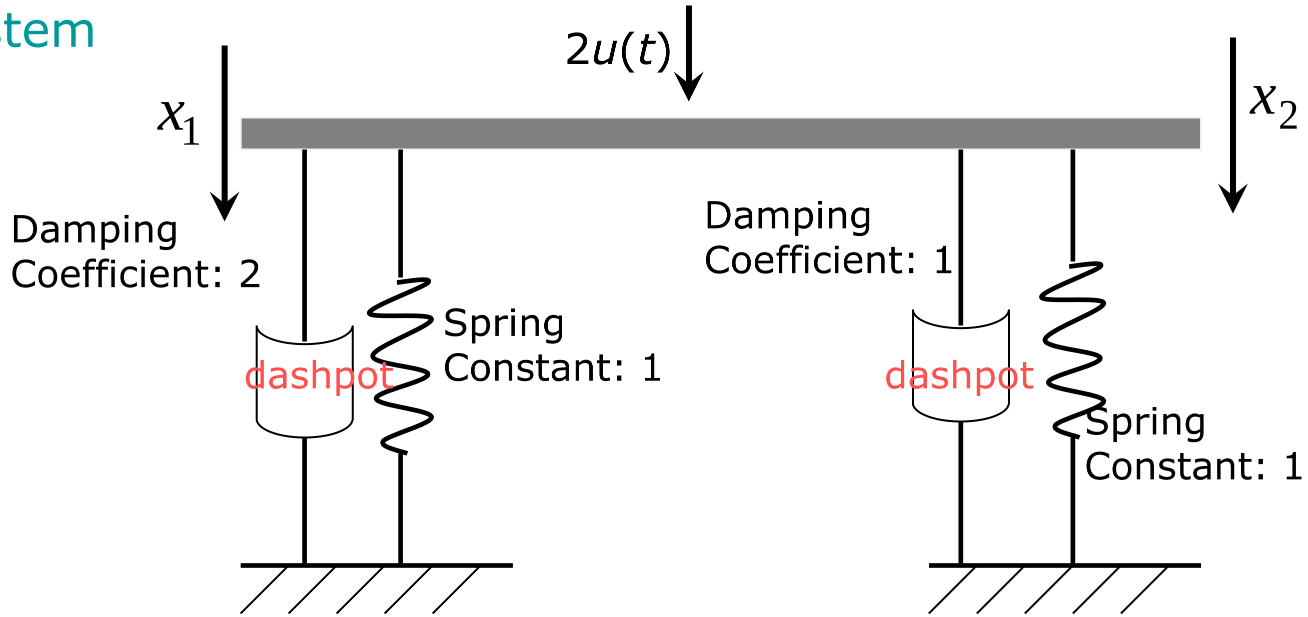
Similarly, X^{contr} is the subspace spanned by the controllable states.

From the property of reachable system, we have

The system is controllable $\Longleftrightarrow$ $\text{rank}\,(R_n(A,B)) = n$

Controllability measure

Example: Platform system



The system is described by the following linear time invariant (LTI) system: assume mass of the platform is zero, and from Newton's law:

$$F - \eta v - kx = ma$$

$$u - 2\dot{x}_1 - x_1 = 0$$
$$u - \dot{x}_2 - x_2 = 0$$

$$\Longrightarrow \quad dx(t)/dt = \underbrace{\begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0.5 \\ 1 \end{pmatrix}}_{B} u(t)$$

Controllability measure

Is the platform system controllable?

The system is controllable $\Longleftrightarrow$ $rank(R_n(A,B)) = n$

$$R_n(A,B) = [B, AB,]$$

$$B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \qquad AB = \begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.25 \\ -1 \end{pmatrix}$$

B, AB are linearly independent!

$$rank(R_n(A,B)) = 2 = n$$

Therefore, the platform system is controllable.

Controllability measure

Associated with controllability, there is the concept of observability.

Controllability: input $u(t)$ $\longrightarrow$ state $x(t)$.

Possibility of steering the state from the input.

Observability: output $y(t)$ $\longrightarrow$ state $x(t)$.

Possibility of estimating the state from the output.

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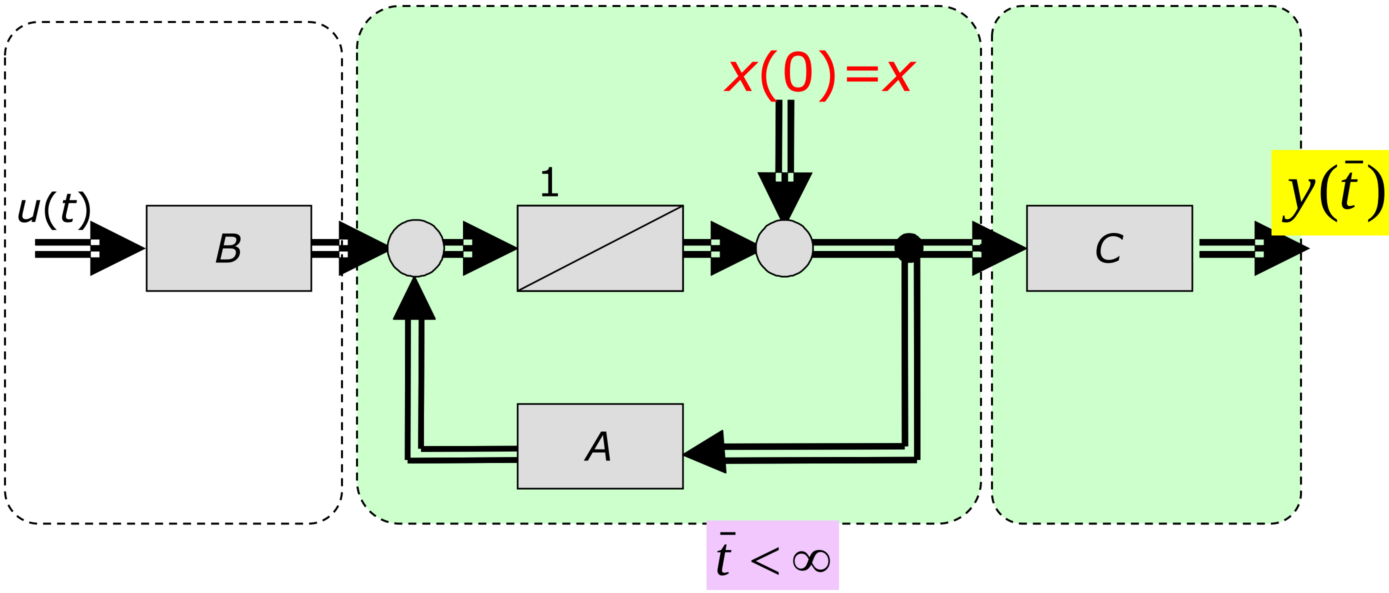
Observability measure

Observability is a measure for how well internal states of a system can be estimated by knowledge of its external outputs.

Definition of Observability: Given any input $u(t)$, a state x of the system is observable, if starting with the state x ($x(0)=x$), and after a finite period of time $\bar{t} < \infty$, x can be uniquely determined by the output $y(\bar{t})$.



Observability measure

Observability matrix?

Observability Gramian?

Output energy?

$$O(L,A) = \begin{pmatrix} L^T \\ L^T A \\ L^T A^2 \\ \vdots \end{pmatrix}$$

Observability measure

Derivation of Observability matrix

From the analytical solution of $dx/dt = Ax + Bu$, we see that after time $\bar{t} < \infty$:

$$\tilde{x}(\bar{t}) = e^{A\bar{t}} x_0 + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau$$

The system starting with $x(0)=x$, therefore

$$\tilde{x}(\bar{t}) = e^{A\bar{t}} x + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau$$

And the output corresponding to $\tilde{x}(\bar{t})$ is:

$$y(\bar{t}) = L^T \tilde{x}(\bar{t}) = L^T e^{A\bar{t}} x + L^T \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau$$

$$= L^T e^{A\bar{t}} x + L^T e^{A\bar{t}} \int_0^{\bar{t}} e^{-A\tau} Bu(\tau) d\tau$$

$$= L^T e^{A\bar{t}} \bar{x} \quad \text{and} \quad \bar{x} = x + \int_0^{\bar{t}} e^{-A\tau} Bu(\tau) d\tau$$

Observability measure

Derivation of Observability matrix

If x is observable, then for any $u(t)$, x can be uniquely determined by the corresponding y :

$$y(\bar{t}) = L^T e^{A\bar{t}} \bar{x} \qquad and \qquad \bar{x} = x + \int_0^{\bar{t}} e^{-A\tau} Bu(\tau) d\tau$$

Since x can be uniquely determined by $\bar{x}$, it is sufficient to prove that $\bar{x}$ can be uniquely determined by $y(\bar{t})$.

Let us see under what condition can $\bar{x}$ be uniquely determined by $y(\bar{t})$?

Observability measure

Derivation of Observability matrix

$$y(\bar{t}) = L^T e^{A\bar{t}} \bar{x}$$

Differentiate the above equation on both sides and get the derivatives at t=0:

$$y(0) = L^T \bar{x}$$

$$y'(0) = L^T A\bar{x}$$

$$y''(0) = L^T A^2 \bar{x}$$

$$\vdots$$

$$y^{(k)}(0) = L^T A^k \bar{x}$$

$$\Longleftrightarrow \quad \begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^k \end{pmatrix} \bar{x} = \begin{pmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(k)}(0) \end{pmatrix} \qquad (\#)$$

(#) has a unique solution $\bar{x}$ if $\begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^k \end{pmatrix}$ has full row rank n.

Observability measure

Derivation of Observability matrix

Denote:

$$Q_k = \begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^k \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(k)}(0) \end{pmatrix} \Longrightarrow \bar{x} = Q_k^{-1} \bar{y}$$

$\bar{x}$ can be uniquely determined, with *k* being at most *n.*

$L^T \in R^{m \times n}$ if *m*>1, then *k*<n, if *m*=1, *k*=n.

Observability measure

Derivation of Observability matrix

Therefore we define

Observability matrix:

$$O(L,A)=\begin{pmatrix} L^T \\ L^T A \\ L^T A^2 \\ \vdots \end{pmatrix}$$

From above analysis, actually the finite Observability matrix is enough to determine observability:

$$O_n(L,A)=\begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^{n-1} \end{pmatrix}$$

Therefore:

The system is observable $\Longleftrightarrow$ $rank(O_n(L,A))=n$

Observability measure

Output energy

The output energy associated with the initial state *x* is:

$$\| y(\bar{t}) \|^2 = \int_0^{\bar{t}} y(t)^* y(t) dt = \int_0^{\bar{t}} \bar{x}^* e^{A^T t} L L^T e^{At} \bar{x} dt$$

$$= \bar{x}^* \int_0^{\bar{t}} e^{A^T t} L L^T e^{At} dt \bar{x}$$

$$= \bar{x}^* Q(\bar{t}) \bar{x}$$

1. Energy of observation produced by an observable state *x*.
2. Observability measure!

Finite Observability Gramian at time $t < \infty$ is defined as:

$$Q(t) = \int_0^t e^{A^T \tau} L L^T e^{A\tau} d\tau, \quad 0 < t < \infty$$

Observability measure

Observability Gramian

Recall the minimal energy to reach a state *x* at time $\bar{t}$ is

$$\| \bar{u} \|^2 = x^* P^{-1}(\bar{t}) x$$

Notice both energies are related to time.

$$\| \bar{u} \|^2 = x^* P^{-1}(\bar{t}) x \qquad \| y(\bar{t}) \|^2 = \bar{x}^* Q(\bar{t}) \bar{x}$$

$$P(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau, \quad 0 < t < \infty \qquad Q(t) = \int_0^t e^{A^T \tau} L L^T e^{A\tau} d\tau, \quad 0 < t < \infty$$

Finite (reachability) controllability Gramian and observability Gramian will be used to derive the infinite Gramians which

1. Make the two measures computable.
2. will be directly used for truncation in MOR.

Outline

- Overlook
- Controllability measures
- Observability measures
- Infinite Gramians
- MOR: Balanced truncation based on infinite Gramians

Infinite Gramians

—— make the two measures computable

Under which condition, $Q(t)$ and $P(t)$ are bounded when time goes to infinity: $t \to \infty$?

$$P(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau, \quad 0 < t < \infty$$

$$Q(t) = \int_0^t e^{A^T\tau} LL^T e^{A\tau} d\tau, \quad 0 < t < \infty$$

Roughly speaking, $Q(t)$ and $P(t)$ can be bounded when $t \to \infty$, if e^{At} is bounded when $t \to \infty$.

Infinite Gramians

——make the two measures computable

e^{At} is bounded if the real parts of all the eigenvalues of *A* are negative.

Why? Let $A = S^{-1}\Lambda S$ be the eigen-decomposition of *A*,

$$e^{At} = e^{S^{-1}\Lambda S t} = S^{-1}e^{\Lambda t}S = S^{-1}e^{\Lambda_{re}t+\Lambda_{im}t}S = S^{-1}e^{\Lambda_{re}t}e^{\Lambda_{im}t}S$$

$$\Lambda_{re} = \begin{pmatrix} \lambda_1^{re} & & & \\ & \lambda_2^{re} & & \\ & & \ddots & \\ & & & \lambda_n^{re} \end{pmatrix} \qquad \Lambda_{im} = \begin{pmatrix} j\lambda_1^{im} & & & \\ & j\lambda_2^{im} & & \\ & & \ddots & \\ & & & j\lambda_n^{im} \end{pmatrix}$$

$\lambda_i = \lambda_i^{re} + \lambda_i^{im}, \quad i = 1,2,\cdots n$ are eigenvalues of *A*.

Infinite Gramians

—— make the two measures computable

$$e^{At} = e^{S^{-1}\Lambda S t} = S^{-1} e^{\Lambda t} S = S^{-1} e^{t\Lambda_{re}} e^{t\Lambda_{im}} S$$

$$e^{t\Lambda_{re}} = \begin{pmatrix} e^{t\lambda_1^{re}} & & & \\ & e^{t\lambda_2^{re}} & & \\ & & \ddots & \\ & & & e^{t\lambda_n^{re}} \end{pmatrix} \xrightarrow[\lambda_i^{re} < 0]{t \to \infty} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$e^{t\Lambda_{im}} = \begin{pmatrix} e^{tj\lambda_1^{im}} & & & \\ & e^{tj\lambda_2^{im}} & & \\ & & \ddots & \\ & & & e^{tj\lambda_n^{im}} \end{pmatrix} \xrightarrow[e^{tj\lambda_i^{im}} = \cos(t\lambda_i^{im}) + j\sin(\lambda_i^{im})]{t \to \infty} \text{bounded}$$

Infinite Gramians

——— make the two measures computable

Therefore, $e^{At} = e^{S^{-1}\Lambda S} = S^{-1}e^{\Lambda}S = S^{-1}e^{t\Lambda_{re}}e^{t\Lambda_{im}}S \to 0$

if the real parts of all the eigenvalues of *A* are negative.

Therefore the follow limits exists if all the eigenvalues of *A* are negative, i.e. if the system is stable:

$$P = \lim_{t\to\infty} P(t) = \lim_{t\to\infty}\int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau = \int_0^\infty e^{At}BB^T e^{A^Tt}dt$$

$$Q = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\int_0^t e^{A^T\tau}LL^T e^{A\tau}d\tau = \int_0^\infty e^{A^Tt}LL^T e^{At}dt$$

where *P* and *Q* are the infinite Gramians (only for stable systems).

Infinite Gramians

—— make the two measures computable

The infinite Gramians:

$$P = \lim_{t\to\infty} P(t) = \lim_{t\to\infty} \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

$$Q = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty} \int_0^t e^{A^T\tau} LL^T e^{A\tau} d\tau = \int_0^\infty e^{A^T t} LL^T e^{At} dt$$

From the property of integral, we have

$$P \geq P(t), \quad \forall t \qquad\qquad Q \geq Q(t), \quad \forall t$$

$$\Downarrow$$

In the meaning of inner product: $P \geq P(t) \Leftrightarrow (Px, x) \geq (P(t)x, x)$

Infinite Gramians

——— make the two measures computable

The minimal energy necessary for reaching a reachable state *x* at time *t* is:

$$\| \bar{u} \|^2 = x^* P^{-1}(t) x$$

For stable systems, lower bound of the minimal energy necessary for reaching a reachable state *x* is:

$$\| \bar{u} \|^2 = x^* P(t)^{-1} x \geq x^* P^{-1} x \quad \text{because} \quad P \geq P(t), \quad \forall t$$

For stable systems, the upper bound of the energy produced by the observable state *x* is:

$$\| y(t) \|^2 = \bar{x}^* Q(t) \bar{x} \leq \bar{x}^* Q \bar{x} \quad \text{because} \quad Q \geq Q(t), \quad \forall t$$

Computable measures!

Only suitable for stable systems!

Infinite Gramians

make the two measures computable

For stable systems, the minimal energy necessary for reaching a state is:

$$\min \| \overline{u} \|^2 = x^* P^{-1} x$$

For stable systems, the maximum energy produced by a state *x* is:

$$\max \| y(t) \|^2 = \overline{x}^* Q \overline{x}$$

Infinite Gramians

make the two measures computable

Because the MOR method we will introduce uses P and Q to derive the reduced-order model, and therefore is only suitable for stable systems.

$$\min \| \bar{u} \|^2 = x^* P^{-1} x \qquad \max \| y(t) \|^2 = \bar{x}^* Q \bar{x}$$

$$\Downarrow$$

The eigenspaces of P and Q make the two measurements practically computable!

Eigenspaces of P and Q

—— make the two measures parctically computable

The states which are difficult to reach are included in the subspace spanned by those eigenvectors of *P* that corresponds to small eigenvalues.

The states which are difficult to observe are included in the subspace spanned by those eigenvectors of *Q* that corresponds to small eigenvalues.

why and how?

Eigenspaces of P and Q

—— make the two measures practically computable

Denote $\xi_1, \xi_2, \cdots, \xi_n$ as the n eigenvectors of *P*, the corresponding eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. (*P* is symmetric, it has real eigenvalues.)

$\xi_1, \xi_2, \cdots, \xi_n$ are linearly independent, therefore they constitute a basis of the whole space C^n.

The state *x* can therefore be represented by $\xi_1, \xi_2, \cdots, \xi_n$:

$$x = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n$$

$$\min \| \bar{u} \|^2 = x^* P^{-1} x$$

If a matrix is nonsingular, then its inverse has the same eigenvectors, but the eigenvalues are the reciprocals:

$$P\xi = \lambda\xi \Rightarrow P^{-1}P\xi = \lambda P^{-1}\xi \Rightarrow \xi / \lambda = P^{-1}\xi$$

Eigenspaces of P and Q

—— make the two measures practically computable

$$\min \| \bar{u} \|^2 = x^* P^{-1} x$$

$$x = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n$$

$$\Downarrow$$

$$P^{-1} x = \alpha_1 \frac{1}{\lambda_1} \xi_1 + \alpha_2 \frac{1}{\lambda_2} \xi_2 + \cdots + \alpha_n \frac{1}{\lambda_n} \xi_n$$

$$\Downarrow$$

$$x^* P^{-1} x = \alpha_1^2 \frac{1}{\lambda_1} \xi_1^* \xi_1 + \alpha_2^2 \frac{1}{\lambda_2} \xi_2^* \xi_2 + \cdots + \alpha_n^2 \frac{1}{\lambda_n} \xi_n^* \xi_n$$

P is symmetric, $\Downarrow$ therefore $\tilde{Q} = [\xi_1, \cdots, \xi_n]$ is orthogonal.

$$\min \| \bar{u} \| = \alpha_1^2 \frac{1}{\lambda_1} + \alpha_2^2 \frac{1}{\lambda_2} + \cdots + \alpha_n^2 \frac{1}{\lambda_n}$$

$\min \| \bar{u} \|^2$ indicates the minimal energy needed to reach the state *x*, therefore the larger $\min \| \bar{u} \|^2$ is, the more difficult the state *x* to reach.

Eigenspaces of P and Q

—— make the two measures practically computable

$$\begin{cases} \min \| \bar{u} \| = \alpha_1^2 \frac{1}{\lambda_1} + \alpha_2^2 \frac{1}{\lambda_2} + \cdots + \alpha_n^2 \frac{1}{\lambda_n} \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \Rightarrow \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \cdots \leq \frac{1}{\lambda_n} \end{cases}$$

$\min \| \bar{u} \|^2$ is larger if $\lambda_1 \geq \lambda_2 \geq \cdots >> \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$ and
$\alpha_1, \alpha_2, \cdots << \alpha_k, \alpha_{k+1}, \cdots, \alpha_n$ than if

$\lambda_1 \geq \lambda_2 \geq \cdots >> \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$ and

$\alpha_1, \alpha_2, \cdots >> \alpha_k, \alpha_{k+1}, \cdots, \alpha_n$

$$x = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n$$

This means if *x* is difficult to reach ($\| \bar{u} \|^2$ is large), *x* should have large components in the subspace spanned by the eigenvectors corresponding to the small eigenvalues of *P*. Or *x* should almost locates in the subspace spanned by the eigenvectors corresponding to the small eigenvalues.

Eigenspaces of P and Q

——make the two measures practically computable

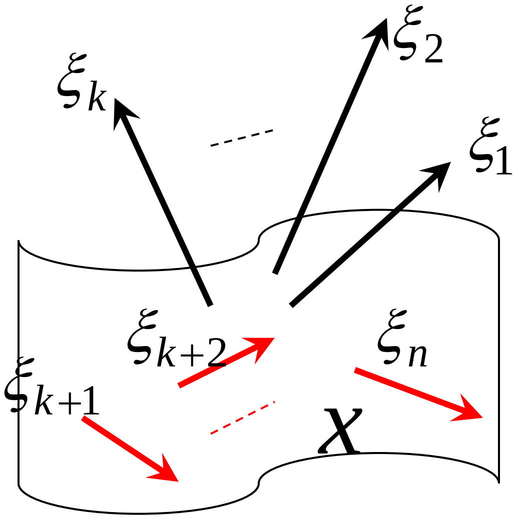
Similarly, if *x* is difficult to observe ($\| y(t) \|^2 = \bar{x}^* Q \bar{x}$ is small) *x* should have large components in the subspace spanned by the eigenvectors corresponding to the small eigenvalues of *Q*. Or *x* should almost locates in the subspace spanned by the eigenvectors corresponding to the small eigenvalues.

$$\lambda_1 \ge \lambda_2 \ge \cdots >> \lambda_k \ge \lambda_{k+1} \ge \cdots \ge \lambda_n$$

$$P\xi_i = \lambda_i \xi_i, i = 1,2,\cdots n$$

$$\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \cdots >> \tilde{\lambda}_k \ge \tilde{\lambda}_{k+1} \ge \cdots \ge \tilde{\lambda}_n$$

$$Q\tilde{\xi}_i = \tilde{\lambda}_i \tilde{\xi}_i, i = 1,2,\cdots n$$



Eigenspaces of P and Q

—— make the two measures practically computable

Till now it seems we could do the truncation by finding subspace spanned by the eigenvectors corresponding to the small eigenvalues of P or Q.

However, it could happen that states which are difficult to reach produce the maximal energy of observation; states which produce the smallest energy of observation are nevertheless the easiest to reach!

For such system, we do not know which states to truncate!

Eigenspaces of P and Q

——— make the two measures practically computable

Example: Consider the following LTI system

$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

$$A = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The two Gramians are: $P = \begin{pmatrix} 2.5 & -1 \\ -1 & 0.5 \end{pmatrix}, Q = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}$

Their eigenvalues and eigenvectors are:

$$\xi_1^P = \begin{pmatrix} 0.92388 \\ -0.38268 \end{pmatrix}, \lambda_1 = 2.91421 \qquad \xi_1^Q = \begin{pmatrix} 0.52573 \\ 0.85865 \end{pmatrix}, \lambda_1^Q = 1.30901$$

$$\xi_2^P = \begin{pmatrix} 0.38268 \\ 0.92388 \end{pmatrix}, \lambda_2^P = 0.08578 \qquad \xi_2^Q = \begin{pmatrix} -0.85865 \\ 0.52573 \end{pmatrix}, \lambda_2^Q = 0.19098$$

Eigenspaces of P and Q

——make the two measures practically computable

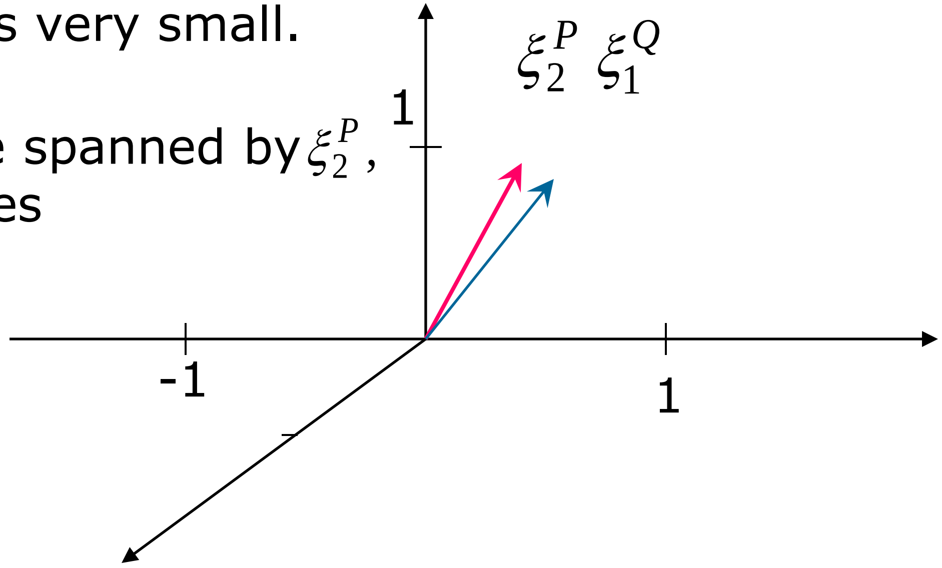
$$\xi_2^P = \begin{pmatrix} 0.38268 \\ 0.92388 \end{pmatrix}, \lambda_2^P = 0.08578 \qquad \xi_1^Q = \begin{pmatrix} 0.52573 \\ 0.85865 \end{pmatrix}, \lambda_1^Q = 1.30901$$

The angle between ξ_2^P, and ξ_1^Q is very small.

This means if S is the subspace spanned by ξ_2^P, then the easily observable states

$$x = \alpha_1 \xi_1^Q + \alpha_2 \xi_2^Q, \alpha_1 >> \alpha_2$$

may also in S.



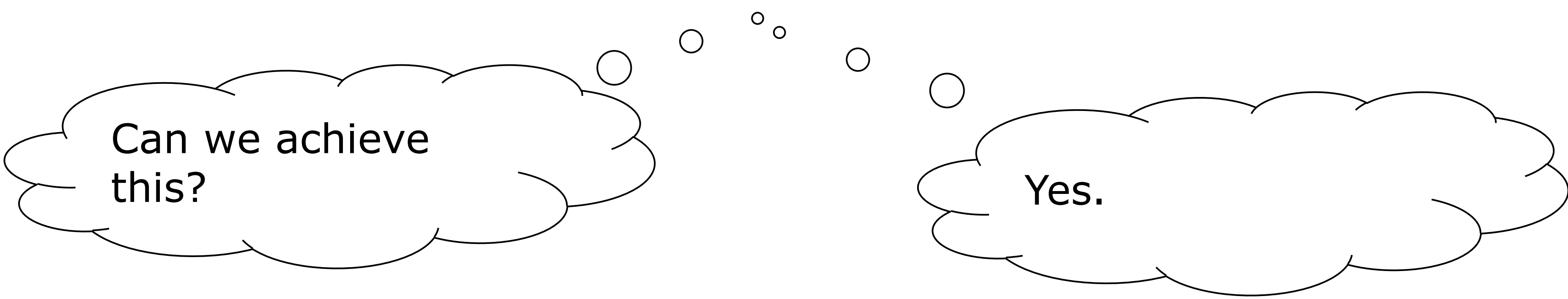
It tells us if we truncate the states which are difficult to reach (the states locate in S), we risk truncating the states which are easy to observe (produce the maximal energy of observation) , because they are also in S).

Eigenspaces of P and Q

——make the two measures practically computable

However, if *P* and *Q* have the same eigenvalues and eigenvectors, then the problems is solved.

The states in the subspace spanned by the eigenvectors of P corresponding to the small eigenvalues always in the subspace spanned by the eigenvectors of Q corresponding to the small eigenvalues, because the eigenvalues are the same and eigenvectors are the same, therefore the subspaces are the same.



We can achieve it by balancing.

Outline

- Overlook
- Controllability measures
- Observability measures
- Infinite Gramians
- MOR: Balanced truncation based on infinite Gramians

MOR: Balanced truncation

Balancing

Recall the Balanced truncation method:

Given a LTI system: $dx(t)/dt = Ax(t) + Bu(t)$

$$y(t) = L^T x(t)$$

$$\left(\begin{array}{c|c} A & B \\ \hline L^T & \end{array}\right) \xrightarrow{\text{balancing}} \left(\begin{array}{c|c} \tilde{A} & \tilde{b} \\ \hline \tilde{L}^T & \end{array}\right) = \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{L}_1^T & \tilde{L}_2^T & \end{array}\right)$$

truncate

$$\text{reduced model} \leftarrow \left(\begin{array}{c|c} \tilde{A}_{11} & \tilde{b}_1 \\ \hline \tilde{L}_1^T & \end{array}\right)$$

The unimportant part is truncated

Basic idea of balancing transformation:

Use state space transformation $\tilde{x} = Tx$ to get another realization of the same system, so that the transformed Gramians are diagonal matrices.

Definition of Balancing transformation:

Finding a nonsingular matrix *T*, such that $\tilde{P} = TPT^T, \tilde{Q} = T^{-T}QT^{-1}$ and $\tilde{P} = \tilde{Q}$.

Definition of Balanced system:

The reachable, observable and stable LTI system is balanced, if its two Gramians are equal $P = Q$, it is principal-axis balanced if

$$P = Q = \Sigma = diag(\sigma_1, \cdots, \sigma_n).$$

MOR: Balanced truncation

——Balancing

Basic idea of balancing transformation:

Use state space transformation $\tilde{x} = Tx$ to get another realization of the same system, so that the transformed Gramians are equal and are diagonal matrices. I.e.

$$\tilde{P} = TPT^T = \Sigma, \quad \tilde{Q} = T^{-T} Q T^{-1} = \Sigma$$

How to construct T?

Recall that $\tilde{P}\tilde{Q} = TPQT^{-1}$.

Since $\tilde{P}\tilde{Q} = \Sigma^2$, we have $TPQT^{-1} = \Sigma^2$, which means $PQ = T^{-1}\Sigma^2 T$.

T should be the inverse of the matrix of eigenvectors of PQ.

MOR: Balanced truncation

——Balancing

Check : $\tilde{P} = TPT^T = ?$ How to make $TPT^T = \Sigma$?

If $P = UU^T$, then $TPT^T = TUU^TT^T = \tilde{T}U^{-1}UU^TU^{-T}\tilde{T}^T = I$ if $\tilde{T}\tilde{T}^T = I$.

Here we must have the relation $T = \tilde{T}U^{-1}$.

If further $T = \Sigma^{1/2}\tilde{T}U^{-1}$, then $TPT^T = \Sigma^{1/2}\tilde{T}U^{-1}UU^TU^{-T}\tilde{T}^T\Sigma^{1/2} = \Sigma$.

It looks that we can compute $\tilde{T}$ as $\tilde{T} = \Sigma^{-1/2}TU$

However, we know that T is the inverse of the eigenvectors of PQ. Since PQ is not a p.s.d. matrix, we have to compute the inverse of the matrix of eigenvectors to get T.

To avoid computing the inverse of the matrix of eigenvectors, we compute $\tilde{T}$ in a different way.

MOR: Balanced truncation

——Balancing

Substitute $T = \Sigma^{1/2}\tilde{T}U^{-1}$ into $TPQT^{-1} = \Sigma^2$, we get

$$\Sigma^{1/2}\tilde{T}U^{-1}PQU\tilde{T}^{-1}\Sigma^{-1/2} = \Sigma^2$$

The left hand side $= \Sigma^{1/2}\tilde{T}U^{-1}UU^TQU\tilde{T}^{-1}\Sigma^{-1/2} = \Sigma^{1/2}\tilde{T}U^TQU\tilde{T}^{-1}\Sigma^{-1/2}$.

Look at the right hand side, we get

$$\Sigma^{1/2}\tilde{T}U^TQU\tilde{T}^{-1}\Sigma^{-1/2} = \Sigma^2,$$

i.e. $\tilde{T}U^TQU\tilde{T}^{-1} = \Sigma^2$. Therefore $\tilde{T}$ is the inverse of the matrix of eigenvectors of U^TQU.

Furtunately, U^TQU is a p.s.d. matrix. therefore the inverse of the matrix of eigenvectors is exactly the transpose of the matrix itself. So that we do not have to compute the inverse.

MOR: Balanced truncation

The above analysis clearly shows that:

Existence of balancing transformation: $dx(t)/dt = Ax(t) + Bu(t)$

$$y(t) = L^T x(t)$$

Given a reachable, observable and stable LTI system and the corresponding Gramians *P* and *Q*, a (principal axis) balancing transformation is given as follows:

$$T = \Sigma^{1/2} K^T U^{-1} \qquad and \qquad T^{-1} = U K^{-T} \Sigma^{-1/2}$$

Here, $P = UU^T$ is the Cholesky factorization of *P*. $U^T QU = K\Sigma^2 K^T$ is the eigen-decomposition of $U^T QU$. (Symmetric positive semi-definite matrix has real non-negtive eigenvalues and orthogonal eigenvectors. Here, the Eigenvectors in *K* are taken as orthonormal)

MOR: Balanced truncation

What is the corresponding balanced system?

Apply the state space tansformation: $\tilde{x} = Tx$ to the original realization:

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) \\ y(t) &= L^T x(t) \end{aligned} \xrightarrow{\tilde{x} = Tx} \begin{aligned} d\tilde{x}(t)/dt &= TAT^{-1}\tilde{x}(t) + TBu(t) \\ y(t) &= L^T T^{-1}\tilde{x}(t) \end{aligned}$$

Balancing :

- Given
$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

- Compute *P*, *Q.*

- Compute $P = UU^T \quad U^T Q U = K\Sigma^2 K^T$

The eigenvalues are ordered from the largest to the smallest

- $$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) \\ y(t) &= L^T x(t) \end{aligned} \xrightarrow[T^{-1} = UK\Sigma^{-1/2}]{T = \Sigma^{1/2} K^T U^{-1}} \begin{aligned} d\tilde{x}(t)/dt &= TAT^{-1}\tilde{x}(t) + TBu(t) \\ y(t) &= L^T T^{-1}\tilde{x}(t) \end{aligned}$$

MOR: Balanced truncation

——Truncate

balanced system: $d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t)$

$$y(t) = L^T T^{-1}\tilde{x}(t)$$

$\tilde{P} = \tilde{Q} = \Sigma \Rightarrow$ the unit vectors e_i are the eigenvectors of $\Sigma : \Sigma e_i = \sigma_i e_i, i = 1, \ldots, n.$

Assume that the elements on the diagonal of Σ is already ordered as : $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$.
Therefore $e_1, \ldots e_r$ span the subspace containing easily controllable and easily observable states.

Truncate the difficult-to-observe and difficult-to-control states means:

$$\tilde{x} = e_1\tilde{x}_1 + \ldots e_n\tilde{x}_n \approx e_1\tilde{x}_1 + \ldots e_r\tilde{x}_r = (\tilde{x}_1, \ldots \tilde{x}_r, 0, \ldots, 0)^T.$$

I.e. $\tilde{x} \approx (\tilde{x}_1, \ldots \tilde{x}_r, 0, \ldots, 0)^T =: x_T$. Replace $\tilde{x}$ with x_T in the balanced system:

$$dx_T(t)/dt = TAT^{-1}x_T(t) + TBu(t)$$
$$y(t) = L^T T^{-1}x_T(t)$$

$$\xrightarrow{z := (\tilde{x}_1, \ldots \tilde{x}_r)}$$

$$d\begin{pmatrix} z(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11}z(t) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 u(t) \\ \tilde{B}_2 u(t) \end{pmatrix}$$
$$y(t) = \begin{pmatrix} \tilde{L}_1^T z(t) & 0 \end{pmatrix}$$

MOR: Balanced truncation

——Truncate

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \qquad \tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \qquad \tilde{L}^T = LT^{-1} = \begin{pmatrix} \tilde{L}_1^T & \tilde{L}_2^T \end{pmatrix}$$

$$d\begin{pmatrix} z(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} z(t) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 u(t) \\ \tilde{B}_2 u(t) \end{pmatrix}$$
$$y(t) = \begin{pmatrix} \tilde{L}_1 z(t) & 0 \end{pmatrix}$$

is a non-minimal realization of a system.

A minimal realization of the same system is:

$$dz(t) = \tilde{A}_{11} z(t) + \tilde{B}_1 u(t)$$
$$\hat{y}(t) = \tilde{L}_1 z(t)$$

The reduced-order model (ROM)

Therefore we have the following simple steps for truncation:

MOR: Balanced truncation

Balancing:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad y(t) = L^T x(t)$$

$$\xrightarrow[T^{-1} = UK\Sigma^{-1/2}]{T = \Sigma^{1/2} K^* U^{-1}}$$

$$d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t), \quad y(t) = L^T T^{-1}\tilde{x}(t)$$

⇓

$$dz(t)/dt = \tilde{A}_{11} z(t) + \tilde{B}_1 u(t)$$
$$\hat{y}(t) = \tilde{L}_1^T z(t)$$

⇓

Reduced model!

Truncate:

$$TPT^T = \Sigma \quad \textit{and} \quad T^{-T} Q T^{-1} = \Sigma$$

$$\Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix}$$

Small part (Σ_2)

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$$

$$\tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}$$

$$\tilde{L}^T = LT^{-1} = \begin{pmatrix} \tilde{L}_1^T & \tilde{L}_2^T \end{pmatrix}$$

Separated according to the separation of Σ.

MOR: Balanced truncation

Balancing:

$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

$$T = \Sigma^{1/2} K^T U^{-1} \quad \longrightarrow \quad T^{-1} = UK\Sigma^{-1/2}$$

$$d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t)$$
$$y(t) = L^T T^{-1}\tilde{x}(t)$$

- Does it make sense if we do model reduction on the balanced system rather than the original system?

  Yes. As a state transformation, balancing does not change the transfer Function, and the HSVs The balanced system is only a different realization of the system.

MOR: Balanced truncation

Observe:

$$dx(t)/dt = Ax(t) + Bu(t) \qquad d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t) \qquad dz(t)/dt = \tilde{A}_{11}z(t) + \tilde{B}_1 u(t)$$
$$y(t) = L^T x(t) \quad \longrightarrow \quad y(t) = L^T T^{-1}\tilde{x}(t) \quad \longrightarrow \quad \hat{y}(t) = \tilde{L}_1^T z(t)$$

$x = T^{-1}\tilde{x} = Y\tilde{x}$. Here the columns in $Y := T^{-1}$ are the eigenvectors of the matrix product PQ. $Y = T^{-1} \Rightarrow T = Y^{-1} =: (W_1 \quad W_2)^T$

If separate Y as $Y = (Y_1, Y_2)$, and $\tilde{x}$ as $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$, then $x = Y\tilde{x} = Y_1\tilde{x}_1 + Y_2\tilde{x}_2$.

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} W_1^T \\ W_2^T \end{pmatrix} A \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} = \begin{pmatrix} W_1^T A Y_1 & W_1^T A Y_2 \\ W_2^T A Y_1 & W_2^T A Y_2 \end{pmatrix} \Rightarrow \tilde{A}_{11} = W_1^T A Y_1$$

$$\tilde{B} = TB = \begin{pmatrix} W_1^T \\ W_2^T \end{pmatrix} B = \begin{pmatrix} W_1^T B \\ W_2^T B \end{pmatrix} \Rightarrow \tilde{B}_1 = W_1^T B \qquad \tilde{L}^T = L^T T^{-1} = L^T \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} = \begin{pmatrix} L^T Y_1 & L^T Y_2 \end{pmatrix} \Rightarrow \tilde{L}_1 = L^T Y_1$$

MOR: Balanced truncation

Therefore the two ROMs are the same:

$$dz(t)/dt = \tilde{A}_{11} z(t) + \tilde{B}_1 u(t)$$
$$\hat{y}(t) = \tilde{L}_1^T z(t)$$

$$d\tilde{x}_1(t)/dt = W_1^T A Y_1 \tilde{x}_1(t) + W_1^T B u(t)$$
$$y(t) = L^T Y_1 \tilde{x}_1(t)$$

Conclusion: balanced truncation is Petrov-Galerkin projection as below:

Let $x \approx Y_1 \tilde{x}_1$

$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

$$\xrightarrow[\text{Petrov-Galerkin using } W_1^T]{x \approx Y_1 \tilde{x}_1}$$

$$d\tilde{x}_1(t)/dt = W_1^T A Y_1 \tilde{x}_1(t) + W_1^T B u(t)$$
$$y(t) = L^T Y_1 \tilde{x}_1(t)$$

Therefore, balanced truncation is equivalent to: finding the invariant subspace of PQ, and remaining only the part (Y_1) which corresponds to the largest HSVs (square root of the eigenvalues of PQ).

MOR: Balanced truncation

Algorithm 1

Given $dx(t)/dt = Ax(t) + Bu(t)$

$y(t) = L^T x(t)$

- Balancing:

1. Compute *P*, *Q*.

2. Compute $P = UU^T \qquad U^T Q U = K\Sigma^2 K^T$

3. $T = \Sigma^{1/2} K^T U^{-1}, \quad T^{-1} = UK\Sigma^{-1/2} \qquad \Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix}$

4. Balancing and separating A, B, L according to the separation of Σ :

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \quad \tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \quad \tilde{L}^T = L^T T^{-1} = \begin{pmatrix} \tilde{L}_1^T & \tilde{L}_2^T \end{pmatrix}$$

- Truncate:

5. Form the reduced model: $d\hat{x}(t)/dt = \tilde{A}_{11}\hat{x}(t) + \tilde{B}_1 u(t)$

$\hat{y}(t) = \tilde{L}_1^T \hat{x}(t)$

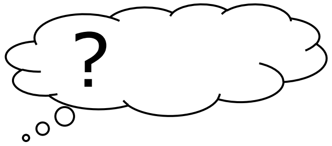
MOR: Balanced truncation

——computational details

Are we ready to get the reduced model from the above Algorithm 1?

Not yet, because we do not know yet how to compute *P* and *Q* numerically!

Recall: $P = \int_0^\infty e^{At} BB^T e^{A^T t} dt \qquad Q = \int_0^\infty e^{A^T t} LL^T e^{At} dt$



Fortunately we have:

Proposition (proposition 4.25 in [Antoulas 05])

P and *Q* are the solution of the following two Lyapunov equations:

$$AP + PA^T = -BB^T$$

$$A^T Q + QA = -LL^T$$

These two matrix equations can be solved numerically (by computer)!
Of course by using some algorithms.

MOR: Balanced truncation

——computational details

In MATLAB, use command:

$$P = lyap(A, B * B^{'})$$

$$Q = lyap(A^T, L * L^{'})$$

MOR: Balanced truncation

—— Numerical issues

The balancing matrix is: $T = \Sigma^{1/2} K^T U^{-1}$, $P = UU^T$.

Computation of U^{-1} may cause numerical instability, because *U* is usually near singular.

U is usually near singular, because the matrix *P* has *numerically* low-rank, i.e. near singular.

P is near singular because in may cases, its eigenvalues decay rapidly to zero, some eigenvalues are very close to zero, e.g. $\lambda_i = 10^{-20}$.

Q and Σ behaves similarly as *P*.

However in algorithm 1, we need to compute:

$$T = \Sigma^{1/2} K^T U^{-1}, \quad T^{-1} = UK\Sigma^{-1/2}$$

Can we avoid computing U^{-1}, Σ^{-1} ?

MOR: Balanced truncation

——computational details

If using Choelsky factorization of both

$$P = Z_P Z_P^T, Q = Z_Q Z_Q^T$$

Observe

$$Z_p^T Q Z_p = Z_p^T Z_Q Z_Q^T Z_p$$

Use SVD instead of eigen- decomposition

$$Z_P^T Z_Q = \tilde{U} \Sigma \tilde{V}^T$$

Comparing with P defined in Algorithm 1, we immediately get

$$T = \Sigma^{1/2} \tilde{U}^T Z_p^{-1}$$

To avoid computing the inverse of Z_p, we have:

$$Z_P^T Z_Q = \tilde{U} \Sigma \tilde{V}^T \Rightarrow Z_p^{-1} = \tilde{U} \Sigma^{-1} \tilde{V}^T Z_Q^T \Longrightarrow T = \Sigma^{-1/2} \tilde{V}^T Z_Q^T$$

MOR: Balanced truncation

——Numerical issues

Algorithm 2 SR method (Getting the reduced model without computing Σ^{-1}, U^{-1}):

1. Do Cholesky factorization of the two Gramians: $P = Z_P Z_P^T, Q = Z_Q Z_Q^T$
   Z_P, Z_Q are lower triangular matrices.
2. Do Singular value decomposition (SVD) of matrix $Z_P^T Z_Q$, i.e., there are two orthonormal matrices $\tilde{U}, \tilde{V}, \quad \tilde{U}^T\tilde{U} = I, \quad \tilde{V}^T\tilde{V} = I$, such that

$$Z_P^T Z_Q = \tilde{U}\Sigma\tilde{V}^T = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix}.$$

3. Let $W = Z_Q \tilde{V}_1 \Sigma_1^{-1/2}, \;\; V = Z_P \tilde{U}_1 \Sigma_1^{-1/2}$.
4. Let $\hat{A} = W^T A V, \hat{B} = W^T B, \hat{L}^T = L^T V$.
5. The reduced model is $d\hat{x}(t)/dt = \hat{A}\hat{x}(t) + \hat{B}u(t)$

$$\hat{y}(t) = \hat{L}^T \hat{x}(t)$$

Do we have done balancing?

MOR: Balanced truncation

Numerical issues

We will prove that the reduced model we got, comes from the above balanced system (balanced by $T=\Sigma^{-1/2}\tilde{V}^T Z_Q^T$)!

The balanced system which is balanced by $T=\Sigma^{-1/2}\tilde{V}^T Z_Q^T$ and $T^{-1}=Z_P\tilde{U}\Sigma^{-1/2}$ is:

$$\tilde{A}=TAT^{-1}=\begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{pmatrix} \qquad \tilde{B}=TB=\begin{pmatrix}\tilde{B}_1\\ \tilde{B}_2\end{pmatrix} \qquad \tilde{L}^T=LT^{-1}=\begin{pmatrix}\tilde{L}_1^T & \tilde{L}_2^T\end{pmatrix}$$

MOR: Balanced truncation

——Numerical issues

The reduced model we obtained just now is:

$$\begin{cases} \hat{A} = W^T A V, \hat{B} = W^T B, \hat{L}^T = L^T V \\ W = Z_Q \tilde{V}_1 \Sigma_1^{-1/2} \quad V = Z_P \tilde{U}_1 \tilde{\Sigma}_1^{-1/2} \\ Z_P^T Z_Q = \tilde{U} \Sigma \tilde{V}^T = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix} \end{cases}$$

$$T = \Sigma^{-1/2} \tilde{V}^T Z_Q^T = \begin{pmatrix} \Sigma_1^{-1/2} \tilde{V}_1^T Z_Q^T \\ \Sigma_2^{-1/2} \tilde{V}_2^T Z_Q^T \end{pmatrix}$$

$$T^{-1} = Z_P \tilde{U} \Sigma^{-1/2} = \begin{pmatrix} Z_P \tilde{U}_1 \Sigma_1^{-1/2} & Z_P \tilde{U}_2 \Sigma_2^{-1/2} \end{pmatrix}$$

Use the block forms above to check if

$$\hat{A} = \tilde{A}_{11}, \hat{B} = \tilde{B}_1, \hat{L}^T = \tilde{L}_1^T$$

Yes, they are equal! The reduced model is really from a balanced system.

MOR: Balanced truncation

—— Numerical issues

Algorithm 2 sometimes cannot continue either, because the Cholesky factorization of P, Q cannot be done. This is because that in some cases P and Q include too small eigenvalues like: $\lambda = 10^{-20}$, which is considered by the algorithm as a singular matrix, therefore Cholesky factorization cannot be continued.

Paper [BennerQ '05] provides an algorithm computing the numerically full rank factors of P and Q , which are in the forms $\tilde{Z}_P \in R^{n\times\hat{n}}, \tilde{Z}_Q \in R^{n\times\hat{n}}$ $\hat{n} << n$

The full rank factors numerically satisfy: $P = \tilde{Z}_P \tilde{Z}_P^T, Q = \tilde{Z}_Q \tilde{Z}_Q^T$

[BennerQ '05] P. Benner, E.S. Quitana-Orti, Model reduction based on spectral projection methods. In: P. Benner, V.L. Mehrmann, D.C. Sorensen (eds.), "Dimension Redution of Large-Scale Systems", vol. 45 of Lecture Notes in Computational Science and Engineering, pp. 5-48, Springer-Verlag, Berlin/Heidelberg, 2005. (**Algorithm 4 in the paper**)

MOR: Balanced truncation

——Numerical issues

Algorithm 3 Getting the reduced model using full-rank factors [BennerQ'05]:

1. Compute full-rank factors of the Gramians: $P = \tilde{Z}_P \tilde{Z}_P^T, Q = \tilde{Z}_Q \tilde{Z}_Q^T$
   $\tilde{Z}_P \in R^{n\times\hat{n}}, \tilde{Z}_Q \in R^{n\times\hat{n}}, \hat{n} << n.$
2. Compute SVD

$$\tilde{Z}_P^T \tilde{Z}_Q = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}.$$

3. Let $W = \tilde{Z}_Q V_1 \Sigma_1^{-1/2}, \;\; V = \tilde{Z}_P U_1 \Sigma_1^{-1/2}.$
4. Let $\hat{A} = W^T A V, \hat{B} = W^T B, \hat{L}^T = L^T V.$
5. The reduced model is $d\hat{x}(t)/dt = \hat{A}\hat{x}(t) + \hat{B}u(t)$

$$\hat{y}(t) = \hat{L}^T \hat{x}(t)$$

MOR: Balanced truncation

——Error bound

Theorem [BennerQ'05]

If the original LTI system is stable, then the reduced model obtained by Algorithm 1, Algorithm 2, Algorithm 3 satisfies:

1) The reduced model is balanced, minimal and stable. It's Gramians are equal to the same diagonal matrix.

2) The absolute error bound (proof in [Antoulas '05] Chapter 7)

$$\| H(s) - \hat{H}(s) \|_{H_\infty} \leq 2 \sum_{k=r+1}^{n} \sigma_k$$

holds.

References

[1] A.C. Antoulas, “Approximation of large-scale Systems”, SIAM Book Series: Advances in Design and Control, 2005.

[2] Chi-Tsong Chen, Linear system Theory and Design, 3rd edition, New York Oxford, Oxford University Press, 1999.