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Computational Methods for Systems and Control Theory

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Model Reduction of Dynamical Systems - 4

Deadline for homework: 10/06/2014

Task: 1 (Reachability Grammian and Reachability Subspace)

Show that the finite reachability gramians $P(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$, $0 < t < \infty$ have the following properties:

- $P(t) = P^T(t) \geq 0$,
- range$(P(t))$ = range$(R(A,B))$, where $R(A,B) = [B,\ AB,\ A^2B \ldots A^{n-1}B \ldots]$
  (The columns of the reachability grammian span the reachability subspace)

Task: 2 (Balancing-free square root (BFSR) method)

We have already discussed the square-root balanced truncation technique for model reduction in Excercise 2, Task 2. Another way of model reduction is to use the balancing-free square root (BFSR) algorithm. Analogue to the standard square-root balanced truncation approach, one has to compute the Cholesky factors S and R of the solutions of the Lyapunov equations and the corresponding SVD of those factors, i.e.,

$$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0, \quad SR^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

The left and right projection matrices for the computation of a reduced-order model of dimension r now are given as $T_l = (Q_1^TP_1)^{-1}Q_1^T$ and $T_r = P_1$, with

$$S^TU_1 = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}, \quad R^TV_1 = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \bar{R} \\ 0 \end{bmatrix},$$

and $P_1, Q_1 \in \mathbb{R}^{n\times r}$ have orthonormal columns and $\hat{R}, \bar{R} \in \mathbb{R}^{r\times r}$ are upper triangular.

Show that the reduced-order system is equivalent to a balanced system and that it satisfies the same error bound as the one obtained by the standard square root BT method.

Task: 3 (Balanced model reduction for non-minimal systems)

Consider a system which is neither controllable nor observable, i.e.,

$$\mathcal{K} = \text{rank}\left(\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}\right) = k_1 < n \quad \text{and} \quad \mathcal{O} = \text{rank}\left(\begin{bmatrix} C \\ CA^T \\ \vdots \\ C(A^T)^{n-1} \end{bmatrix}\right) = k_2 < n.$$

Show that if $Y \in \mathbb{R}^{n\times k_1}$ and $Z \in \mathbb{R}^{n\times k_2}$ are low rank factors that satisfy $P = YY^T$ and $Q = ZZ^T$, a balanced reduced-order model can be obtained by projection matrices $T_l = \Sigma_1^{-\frac{1}{2}}V_1Z^T$ and $T_r = YU_1\Sigma_1^{-\frac{1}{2}}$, where $Y^TZ = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.

Task: 4 (Model reduction by balanced truncation)

Implement the method of balanced truncation introduced in the course. If you do not have access to the control system toolbox, you can use the routine *lyap_sgn_fac.m* from the course homepage to compute approximations to the Cholesky factors of the solutions to the Lyapunov equations. Try your program by means of the model of a beam which you find as *beam.mat* on the course homepage. Evaluate the transfer function

$$H(i\omega) = C(i\omega I - A)^{-1}B$$

for original and reduced-order model over the frequency interval $\omega \in [10^{-2}, 10^{4}]$. Use 10 000 logarithmically distributed sample points. Plot the *gain* of the transfer function, i.e. $20 \cdot \log_{10} |(H(j\omega)|$ on a logarithmic x-scale by using the MATLAB command *semilogx(*ω, H_ω*)*.

Send your routines to *imahmad@mpi-magdeburg.mpg.de*. The filename should include your name and the corresponding exercise sheet number as well as the exercise number, e.g., name-hw1e5. In case of several files please hand in a compressed file. Moreover, please print the source code of your routine and hand it in together with the other exercises.