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Computational Methods for Systems and Control Theory

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Model Reduction of Dynamical Systems - 2

Deadline for homework: 06/05/2014

Task: 1 (Computation of system norms)

a) Consider the LTI system from Excercise 1, Task 4:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; d = 0.$$

Analytically compute the $\mathcal{H}_\infty$-norm of the system.

b) Consider the following LTI system:

$$A = \begin{bmatrix} -8 & 8 \\ -8 & -42 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; d = 0.$$

Analytically compute the $\mathcal{H}_2$-norm of the system.

Hint: Make use of the eigenvalue decomposition of $A = Q\Lambda Q^{-1}$ and the fact that the $\mathcal{H}_2$-norm is invariant under state-space transformations. Further use that the $\mathcal{H}_2$-norm is given as $\sqrt{c^T P c}$, where P satisfies the Lyapunov equation in Exercise 2, Task 2.

Task: 2 (Balanced realizations)

Given a minimal LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0,$$
$$y(t) = Cx(t) + Du(t).$$

Show that a balanced realization is given by the state-space transformation

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$ and $Q = R^T R$ (e.g., Cholesky decompositions) satisfy the pair of Lyapunov equations

$$AP + PA^T + BB^T = 0,$$
$$A^T Q + QA + C^T C = 0$$

and

$$SR^T = U\Sigma V^T$$

is the SVD of SR^T.

Hint: First note that $T^{-1} = S^T U \Sigma^{-\frac{1}{2}}$, then the result follows by simple algebraic manipulations.

Task: 3 (The Lyapunov equation)

Consider the *(infinite) controllability Gramian* $P := \int_0^\infty e^{As}BB^Te^{A^Ts}\,ds$. Assume that (A,B) is controllable. Show that the following two statements are equivalent:

a) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is asymptotically stable.

b) It holds $P > 0$ and $AP + PA^T + BB^T = 0$.

Hint b) $\Rightarrow$ a): Consider an eigenvalue λ of A together with its corresponding eigenvector x. Then pre- and postmultiply the above equation by x^* and x, respectively.

Task: 4 (Minimal balanced realization)

Consider the following LTI system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 5 & -7 & 0 & -2 \\ 6 & -8 & 0 & -2 \\ 0 & 0 & -3 & 0 \\ 9 & -9 & 0 & -4 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}}_{b} u(t),$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}}_{c} x(t).$$

Use the MATLAB command *lyapchol* to compute the Cholesky factors S and R of the solutions to the Lyapunov equations

$$AP + PA^T + bb^T = 0, \quad A^TQ + QA + c^Tc = 0.$$

In case that you do not have access to the Control System toolbox, you can find the results in the file *LyapSol.mat* on the course homepage. Compute the singular value decomposition $U\Sigma V^T = SR^T$ of the product of the Cholesy factors S and R^T. What can you say about the minimality of the system? Modify the procedure from Exercise 2, Task 2 and adjust the transformation matrices $T = \Sigma^{-\frac{1}{2}}V^TR$ and $T^{-1} = S^TU\Sigma^{-\frac{1}{2}}$. Use your results to construct a minimal reduced-order model by an oblique projection which exactly reproduces the transfer function of the original model, i.e., for the reduced system it should hold $\hat{H}(iw) = H(iw)$, $\forall w$. Validate your results by means of plotting the gain of the original and the reduced transfer function with the frequency interval $w \in [10^{-1}, 10^5]$. Use 1000 logarithmically distributed sample points and plot the gain $20 \cdot log_{10}|H(jw)|$ on a logarithmic x-scale by using MATLAB command *semilogx*(w, H_w).

Send your routines to *imahmad@mpi-magdeburg.mpg.de*. The filename should include your name and the corresponding exercise sheet number as well as the exercise number, e.g., name-hw1e5. In case of several files please hand in a compressed file. Moreover, please print the source code of your routine and hand it in together with the other exercises.