Otto-Von-Guericke-University-Magdeburg
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Computational Methods for Systems and Control Theory

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Model Reduction of Dynamical Systems - 1

Deadline for homework: 22/04/2014

Excercise: 1 (Properties of Singular Value Decomposition (SVD))

Let $r = \text{rank}(A)$, $A = U\Sigma V^T$ be the singular value decomposition with orthonormal matrices $U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \in \mathbb{R}^{m\times m}$ and $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n\times n}$. Show the following statements:

a) Schmidt-Eckart-Young-Mirsky-Theorem: If $k < r$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, then it holds:

$$\min_{\text{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1},$$

that means A_k is the best rank-k approximation of A.

Hint: Realize that $\text{rank}(A_k) = k$. Then select a matrix $B \in \mathbb{R}^{m\times m}$ with $\text{rank}(B) = k$. There exist orthonormal vectors $x_1, \ldots, x_{n-k}$ such that $ker(B) = \text{span}\{x_1, \ldots, x_{n-k}\}$ and it holds: $\text{span}\{x_1, \ldots, x_{n-k}\} \cap \text{span}\{v_1, \ldots, v_{k+1}\} \neq \{\emptyset\}$. Let z be a vector of this intersection with $\|z\| = 1$. Then it holds $Bz = 0$. Making use of the representation of Az one can show the necessary estimate.

b) It holds:

$$\ker(A) = \{v_{r+1}, \ldots, v_n\},$$
$$\text{range}(A) = \{u_1, \ldots, u_r\}.$$

b) It holds:

$$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$$

Excercise: 2 Image data compression via SVD

Write a MATLAB routine which reads in a matrix containing image data, computes a best rank-k approximation of this matrix and then displays the approximation (w.r.t. the 2-norm), original and compressed image as well as required memory for storing the images. Try your program by means of a picture of the Cathedral of Magdeburg which you can find as *dom_md_c.mat* on the course homepage. Test different values of k and empirically determine the smallest value of k which can be used without a visible loss of accuracy.

Hint: The MATLAB image is created with the *load* command, e.g.,

```
>> load dom_md_c.mat
```

yields an image data matrix of dimension 200×300. For a visualization of the image in MATLAB, you can use the commands

```
>> colormap(map)
>> image(A)
```

Excercise: 3 (Controllability of dynamical systems)

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $K = [B, AB, \dots, A^{n-1}B]$ be the associated Kalman controllability matrix and let $r := \operatorname{rank}(K)$.

a) Show that there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad \tilde{B} := T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where (A_1, B_1) is a controllable matrix pair.

b) Show that (A, B) is controllable iff $[\lambda I - A, B]$ has full row rank $\forall \lambda \in \mathbb{C}$.

Excercise: 4 (Properties of a dynamical system)

Consider the dynamical system given by

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Analyze the above system w.r.t. stability, controllability and observability. Further, compute the transfer function $H(s) = c^T(sI - A)^{-1}b$ of the system. What can you say with regard to minimality of the system?

Excercise: 5 (Minimality of LTI systems)

Consider an LTI system as in theoretical exercise 4. Show that the realization (A, B, C, D) is minimal if and only if (A, B) is controllable and (A, C) is observable.

Hint: W.l.o.g. rewrite the system by means of the Kalman controllability decomposition shown in Exercise sheet 1. For observability of (A, C) make use of the Kalman observability decomposition

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \quad \text{and} \quad \tilde{C} := CT = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$

which yields an observable matrix pair (A_1, C_1).

Send your routines to *imahmad@mpi-magdeburg.mpg.de*. The filename should include your name and the corresponding exercise sheet number as well as the exercise number, e.g., name-hw1e5. In case of several files please hand in a compressed file. Moreover, please print the source code of your routine and hand it in together with the other exercises.