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MPI, Magdeburg

Mathematical Basics

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Outline

Numerical Linear Algebra

Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory (in single precision): $4 \cdot n_x \cdot n_y$ bytes.

Theorem (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$X \approx \widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Numerical Linear Algebra

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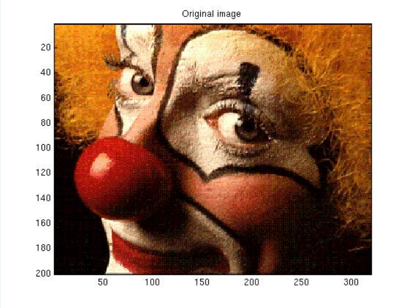
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel

$\rightsquigarrow \quad \approx 256$ kB

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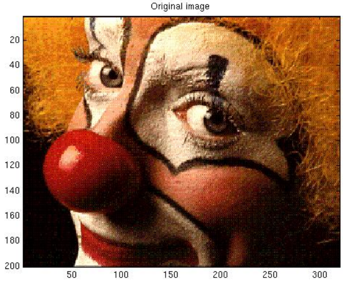

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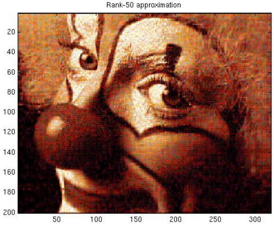
- rank $r = 50$, ≈ 104 kB



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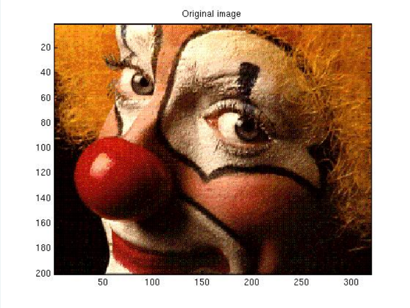

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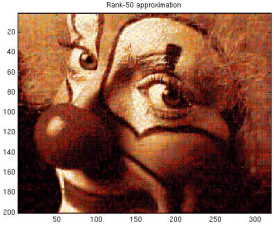
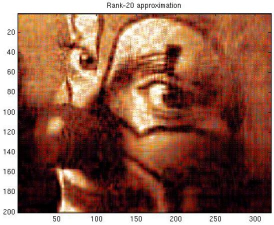


- rank $r = 20$, ≈ 42 kB



Dimension Reduction via SVD

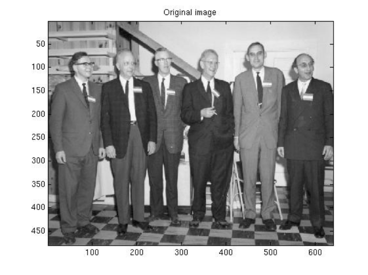
Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens,

George Forsythe, Alston Householder,

Peter Henrici, Fritz L. Bauer.



640×480 pixel, ≈ 1229 kB

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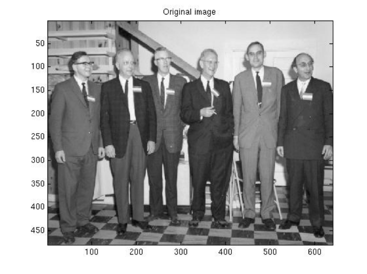
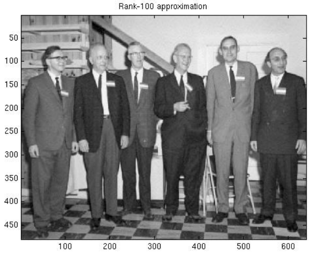
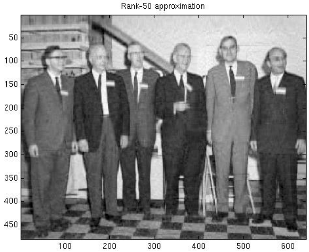


640×480 pixel, ≈ 1229 kB

- rank $r = 100$, ≈ 448 kB



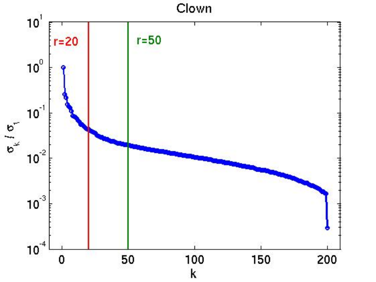
- rank $r = 50$, ≈ 224 kB

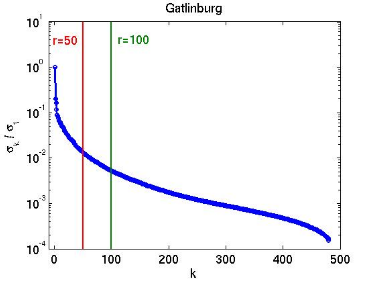


Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices





Systems and Control Theory
The Laplace transform

Definition

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f(t) \mapsto f(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

F is a function in the (Laplace or) frequency domain.

Note: for frequency domain evaluations ("frequency response analysis"), one takes $\mathrm{re}\, s = 0$ and $\mathrm{im}\, s \geq 0$. Then $\omega := \mathrm{im}\, s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi v$ with v measured in [Hz]).

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Systems and Control Theory
The Laplace transform

Lemma

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s) - f(0).$$

if f(0)=0, then

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s).$$

Note: For ease of notation, in the following we will use lower-case letters for both, a function $f(t)$ and its Laplace transform $F(s)$!

Systems and Control Theory

The Model Reduction Problem as Approximation Problem in Frequency Domain

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s))$ to linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sEx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

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$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sE - A)^{-1}B + D}_{=:G(s)} \Big) u(s).$$

$G(s)$ is the transfer function of Σ.

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Goal: Fast evaluation of mapping $u \rightarrow y$.

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Systems and Control Theory

The Model Reduction Problem as Approximation Problem in Frequency Domain

Formulating model reduction in time domain

Approximate the dynamical system

$$\begin{array}{rcll} E\dot{x} & = & Ax + Bu, & E, A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y & = & Cx + Du, & C \in \mathbb{R}^{q\times n},\ D \in \mathbb{R}^{q\times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcll} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, & \hat{E}, \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{q\times r},\ \hat{D} \in \mathbb{R}^{q\times m} \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.$$

Systems and Control Theory
Properties of linear systems

Definition

A linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the eigenvalues of the generalized eigenvalue problem $Ax = \lambda Ex$, denoted by $\Lambda(A, E)$, satisfies $\Lambda(A, E) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

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Systems and Control Theory
Properties of linear systems

Further properties:

- Controllability/reachability
- Observability

will be discussed in the lecture on balanced truncation MOR method. For

- Stabilizability
- Detectability

See handout "Mathematical Basics".

Systems and Control Theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{cases} \qquad \begin{array}{l} \text{with transfer function} \\ G(s) = C(sI - A)^{-1}B + D, \end{array}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a realization of Σ.

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x \to Tx, \\ (A, B, C, D) \to (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

Systems and Control Theory

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Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_2 \in \mathbb{R}^{q \times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Systems and Control Theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

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Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

Systems and Control Theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

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Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Systems and Control Theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

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Theorem

A realization (A, B, C, D) of a linear system is minimal $\iff$
(A, B) is controllable and (A, C) is observable.

Systems and Control Theory
Balanced Realizations

Infinite Gramians

$$P = \int_0^\infty e^{At} B B^T e^{A^T t} dt.$$

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

Systems and Control Theory
Balanced Realizations

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

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When does a balanced realization exist?

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U\Sigma V^T$ is the SVD of SR^T.

Proof. Exercise!

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

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Balanced Realizations

Theorem

The infinite controllability/observability Gramians P/Q satisfy the Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0.$$

Proof. (For controllability Gramian only, observability case is analogous!)

$$\begin{aligned}
AP + PA^T + BB^T &= A\int_0^\infty e^{At}BB^Te^{A^Tt}dt + \int_0^\infty e^{At}BB^Te^{A^Tt}dt\,A^T + BB^T \\
&= \int_0^\infty \underbrace{Ae^{At}BB^Te^{A^Tt} + e^{At}BB^Te^{A^Tt}A^T}_{=\frac{d}{dt}e^{At}BB^Te^{A^Tt}} dt + BB^T \\
&= \underbrace{\lim_{t\to\infty} e^{At}BB^Te^{A^Tt}}_{=0} - \underbrace{e^{A\cdot 0}}_{=I_n}BB^T\underbrace{e^{A^T\cdot 0}}_{=I_n} + BB^T \\
&= 0.
\end{aligned}$$

Systems and Control Theory
Balanced Realizations

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A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \dots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \dots, n-1).$$

$\sigma_1, \dots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \dots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \dots, \sigma_n > 0$ in case of minimality!

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

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Proof. In balanced coordinates, the HSVs are $\Lambda(PQ)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

This is equivalent to

$$0 = A(T^{-1}\hat{P}T^{-T}) + (T^{-1}\hat{P}T^{-T})A^T + BB^T.$$

The uniqueness of the solution of the Lyapunov equation (for stable systems) implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda(\hat{P}\hat{Q}) = \Lambda(PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

Systems and Control Theory
Balanced Realizations

Definition

A realization (A, B, C, D) of a stable linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

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Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\mathrm{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Qualitative and Quantitative Study of the Approximation Error
System Norms

Definition

The $L_2^n(-\infty, +\infty)$ space is the vector-valued function space $f : \mathbb{R} \mapsto \mathbb{R}^n$, with the norm

$$\|f\|_{L_2^n} = \left(\int_{-\infty}^{\infty} ||f(t)||^2 dt \right)^{1/2}.$$

Here and below, $||\cdot||$ denotes the Euclidean vector or spectral matrix norm.

Definition

The frequency domain $\mathcal{L}_2(\jmath\mathbb{R})$ space is the matrix-valued function space $F : \mathbb{C} \mapsto \mathbb{C}^{p\times m}$, with the norm

$$||F||_{\mathcal{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} ||F(\jmath\omega)||^2 d\omega \right)^{1/2},$$

where $\jmath = \sqrt{-1}$ is the imaginary unit.

Qualitative and Quantitative Study of the Approximation Error
System Norms

The maximum modulus theorem will be used repeatedly.

Theorem

Let $f(z) : \mathbb{C}^n \mapsto \mathbb{C}$ be a regular analytic, or holomorphic, function of n complex variables $z = (z_1, \ldots, z_n), n \geq 1$, defined on an (open) domain $\mathbb{D}$ of the complex space $\mathbb{C}^n$, which is not a constant, $f(z) \neq$ const. Let

$$max_f = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

If $f(z)$ is continuous in a finite closed domain $\mathbb{D}$, then max_f can only be attained on the boundary of $\mathbb{D}$.

Qualitative and Quantitative Study of the Approximation Error
System Norms

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If $f(z)$ is continuous in a finite closed domain $\mathbb{D}$, then max_f can only be attained on the boundary of $\mathbb{D}$.

Consider the transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

and input functions $u \in \mathcal{L}_2(\jmath\mathbb{R})$, with the $\mathcal{L}_2$-norm

$$||u||_{\mathcal{L}_2}^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega.$$

Qualitative and Quantitative Study of the Approximation Error
System Norms

Assume A is (asymptotically) stable: $\Lambda(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Re}\,(z) < 0\}$. Then G is analytic in $\mathbb{C}^+ \cup \jmath\mathbb{R}$, and following the maximal modulus theorem, $G(s)$ is bounded: $||G(s)|| \leq M < \infty$, $\forall s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$. Thus we have

$$\begin{aligned}
\int_{-\infty}^{\infty} y(\jmath\omega)^H y(\jmath\omega)\, d\omega &= \int_{-\infty}^{\infty} u(\jmath\omega)^H G(\jmath\omega)^H G(\jmath\omega) u(\jmath\omega)\, d\omega \\
&= \int_{-\infty}^{\infty} ||G(\jmath\omega)u(\jmath\omega)||^2\, d\omega \leq \int_{-\infty}^{\infty} M^2 ||u(\jmath\omega)||^2 d\omega \\
&= M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega \ < \ \infty,
\end{aligned}$$

So that $y = Gu \in \mathcal{L}_2(\jmath\mathbb{R})$.
Consequently, the $\mathcal{L}_2$-induced operator norm

$$||G||_{\mathcal{L}_\infty} := \sup_{||u||_2 \neq 0} \frac{||Gu||_{\mathcal{L}_2}}{||u||_{\mathcal{L}_2}} \tag{1}$$

is well defined [Antoulas '05].

Qualitative and Quantitative Study of the Approximation Error
System Norms

Error bound 1

$$||Gu||_{\mathcal{L}_2} \leq ||G||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_2}$$

Consequently,

$$||y - \hat{y}||_{\mathcal{L}_2} = ||Gu - \hat{G}u||_{\mathcal{L}_2} \leq ||G - \hat{G}||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_2}$$

Qualitative and Quantitative Study of the Approximation Error
System Norms

It can be further proved that

$$||G||_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} ||G(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(G(\jmath\omega)\right).$$

With the above defined $\mathcal{L}_\infty$-norm, the frequency domain $\mathcal{L}_\infty$ space is defined as

Definition

The frequency domain $\mathcal{L}_\infty(\jmath\mathbb{R})$ space is the matrix-valued function space $F : \mathbb{C} \mapsto \mathbb{C}^{p\times m}$, with the norm

$$||F||_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} ||F(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

Qualitative and Quantitative Study of the Approximation Error
System Norms

Definition

The Hardy space $\mathcal{H}_\infty$ is the function space of matrix-, scalar-valued functions that are analytic and bounded in $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$.

The $\mathcal{H}_\infty$-norm is defined as

$$||F||_{\mathcal{H}_\infty} := \sup_{z \in \mathbb{C}^+} ||F(z)|| = \sup_{\omega \in \mathbb{R}} ||F(\jmath\omega)|| = \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

The second equality follows the maximum modulus theorem.

Qualitative and Quantitative Study of the Approximation Error
System Norms

Definition

The Hardy space $\mathcal{H}_2(\mathbb{C}^+)$ is the function space of matrix-, scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm defined as

$$\begin{aligned} ||F||_2 &:= \frac{1}{2\pi}\left(\sup_{\mathrm{re}\sigma>0}\int_{-\infty}^{\infty}||F(\sigma+\jmath\omega)||_F^2 d\omega\right)^{\frac{1}{2}} \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}||F(\jmath\omega)||_F^2\, d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

The last equality follows maximum modulus theorem.

Theorem

Practical Computation of the $\mathcal{H}_2$-norm follows

$$||F||_2^2 = \mathrm{tr}(B^T Q B) = \mathrm{tr}(C P C^T),$$

where P, Q are the controllability and observability Gramians (the infinite Gramians) of the corresponding LTI system.

Qualitative and Quantitative Study of the Approximation Error
System Norms

Following [Antoulas, Beattie, Gugercin '10][1] (pp. 15-16), the $\mathcal{H}_2$ approximation error gives the following bound

$$\max_{t>0} \|y(t) - \hat{y}(t)\|_\infty \le \|G - \hat{G}\|_{\mathcal{H}_2},$$

where G and $\hat{G}$ are original and reduced transfer functions. $||\cdot||_\infty$ is the vector norm in Euclidean space for any fixed t.
and

Error bound 2

$$\|y - \hat{y}\|_\infty \le ||G - \hat{G}||_{\mathcal{H}_2} ||u||_{\mathcal{H}_2}.$$

[1] A. C. Antoulas, C. A. Beattie, S. Gugercin. Interpolatory Model Reduction of Large-scale Dynamical Systems. J. Mohammadpour and K.M. Grigoriadis, *Efficient Modeling and Control, 3 of Large-Scale Systems*, 3-58, Springer Science+Business Media, LLC 2010.

Qualitative and Quantitative Study of the Approximation Error
System Norms

(Plancherel Theorem)

The Fourier transform of $f \in L_2^n(-\infty, \infty)$:

$$F(\xi) = \int_{-\infty}^{\infty} f(t) e^{-\xi t} dt$$

is a Hilbert space isomorphism between $L_2^n(-\infty, \infty)$ and $\mathcal{L}_2(\jmath\mathbb{R})$. Furthermore, the Fourier transform maps $L_2^n(0, \infty)$ onto $\mathcal{H}_2(\mathbb{C}^+)$. In addition it is an isometry, that is, it preserves distances:

$$L_2^n(-\infty, \infty) \cong \mathcal{L}_2(\jmath\mathbb{R}), \quad L_2^n(0, \infty) \cong \mathcal{H}_2(\mathbb{C}^+).$$

Consequently, L_2^n-norm in time domain and $\mathcal{L}_2$-norm, $\mathcal{H}_2$-norm in frequency domain coincide.

Qualitative and Quantitative Study of the Approximation Error
Approximation Problems

Therefore the Error bound 1,

$$||y - \hat{y}||_2 = ||Gu - \hat{G}u||_2 \leq ||G - \hat{G}||_{\mathcal{L}_\infty}||u||_2, \tag{2}$$

holds in time and frequency domain due to Plancherel theorem, i.e. the $||\cdot||_2$ in (2) can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.

The transfer function is analytic, therefore

$$||G||_{\mathcal{L}_\infty} = ||G||_{\mathcal{H}_\infty},$$

so that,

$$||y - \hat{y}||_2 \leq ||G - \hat{G}||_{\mathcal{H}_\infty}||u||_2.$$

Qualitative and Quantitative Study of the Approximation Error
Approximation Problems

Finally, we get two error bounds,

Output errors bounds

$$\begin{array}{rcll} \|y - \hat{y}\|_2 & \leq & ||G - \hat{G}||_{\mathcal{H}_\infty} \|u\|_2 & \Longrightarrow \|G - \hat{G}\|_\infty < \mathrm{tol} \\ \|y - \hat{y}\|_\infty & \leq & ||G - \hat{G}||_{\mathcal{H}_2} \|u\|_2 & \Longrightarrow ||G - \hat{G}||_{\mathcal{H}_2} < \mathrm{tol} \end{array}$$

Goal: $||G - \hat{G}||_\infty < tol$ (2) or $||G - \hat{G}||_{\mathcal{H}_2} < tol$ (**??**).

Qualitative and Quantitative Study of the Approximation Error
Approximation Problems

Finally, we get two error bounds,

Output errors bounds

$$\begin{aligned} \|y-\hat{y}\|_2 &\leq ||G-\hat{G}||_{\mathcal{H}_\infty}\|u\|_2 && \Longrightarrow \|G-\hat{G}\|_\infty < \mathrm{tol} \\ \|y-\hat{y}\|_\infty &\leq ||G-\hat{G}||_{\mathcal{H}_2}\|u\|_2 && \Longrightarrow ||G-\hat{G}||_{\mathcal{H}_2} < \mathrm{tol} \end{aligned}$$

Goal: $||G-\hat{G}||_\infty < tol$ (2) or $||G-\hat{G}||_{\mathcal{H}_2} < tol$ (**??**).

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)

Qualitative and Quantitative Study of the Approximation Error

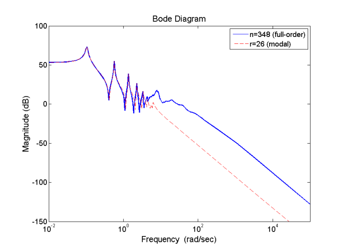
Computable error measures

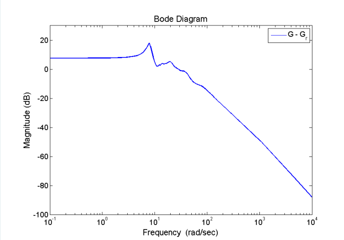
Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ ($j = 1, \ldots, N_\omega$);
- relative errors $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).

  For MIMO systems, $q \times m$ array of plots G_{ij}.





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