Model reduction of dynamical systems
Introduction

Lihong Feng

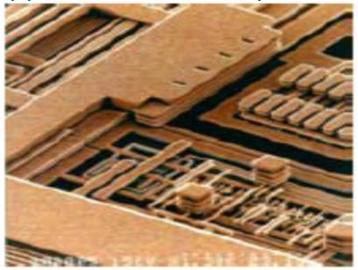
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

Motivation

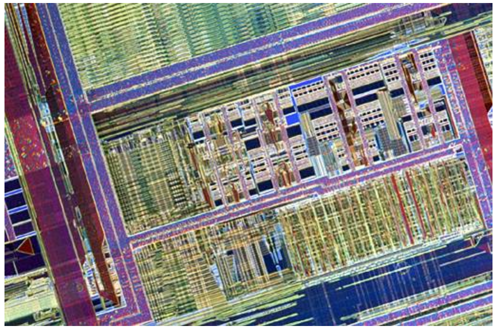
Large-scale dynamical systems are omnipresent in science and technology.

Example 1.

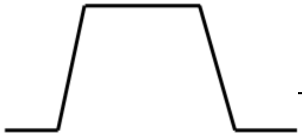
Copper interconnect pattern(IBM)



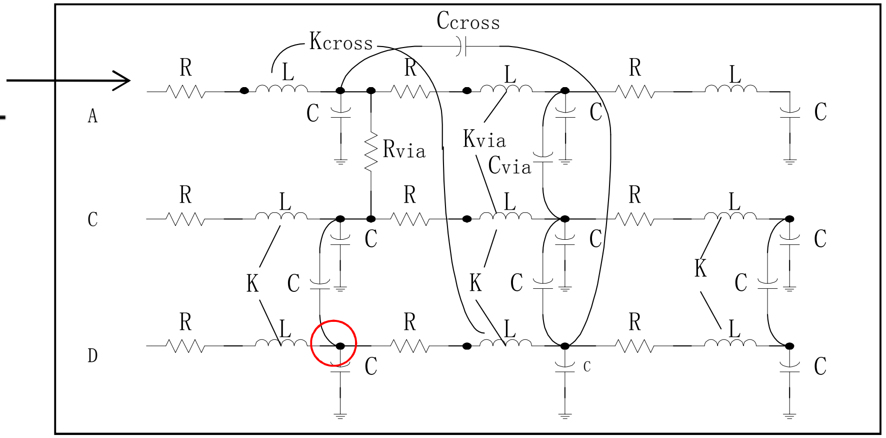
Picture from [N.P can der Meijs'01]



Picture from Encarta
http://encarta.msn.com/media_461519585/pentium_microprocessor.html



Signal integrity:
noise, delay.



Motivation

The interconnect can be modelled by a mathematical model:

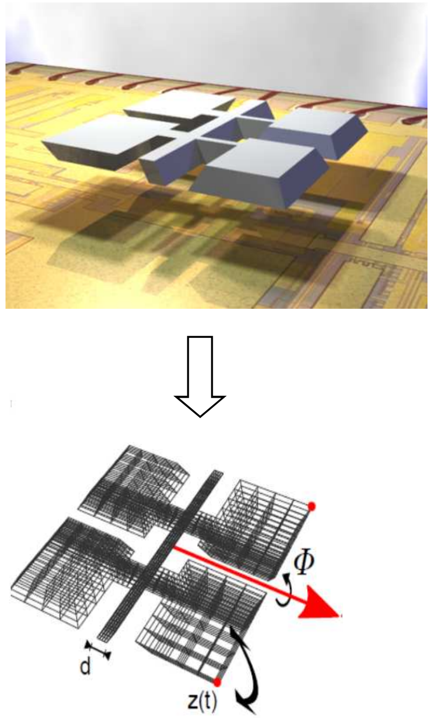
$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

With $O(10^6)$ ordinary equations.

Example

- Gyroscope is a device for measuring or maintaining orientation and has in various automobiles (aviation, shipping and defense).

- The design of the device is verified by modelling and simulation.



Pic. by Jan Lienemann and C. Moosmann, IMTEK

Motivation

- The butterfly Gyroscope can be modeled by partial differential equations (PDEs).

- Using finite element method, the PDE is discretized into (in space) ODEs:

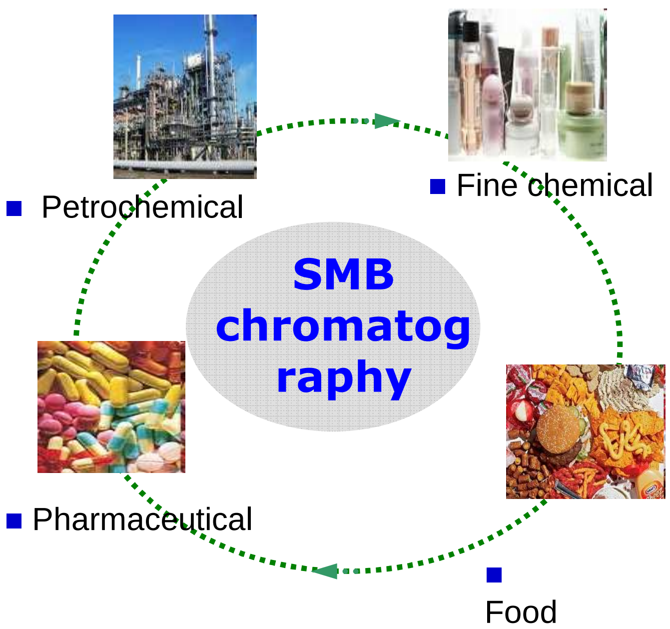
$$M(\mu)\frac{d^2x}{dt^2} + K(\mu)\frac{dx(t)}{dt} + D(\mu)x(t) + Bu(t),$$
$$y(t) = Cx(t),$$

With $O(10^5)$ ordinary differential equations. $\mu = (\mu_1, \dots, \mu_l)$ is the vector of parameters.

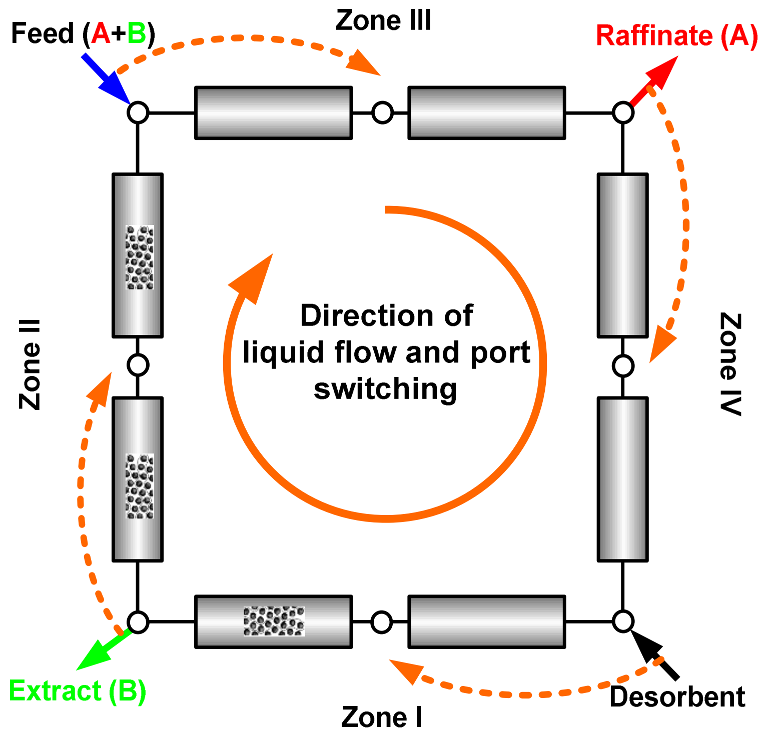
Multi-query, real-time simulation?

Motivation

Example 3 Simulated Moving Bed

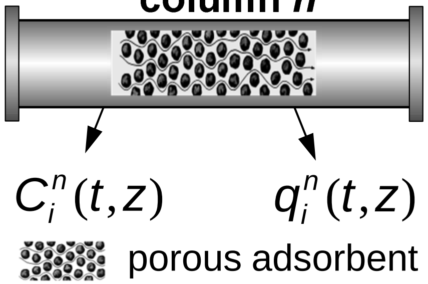


Simulated **M**oving **B**ed Process



Mathematical modeling of SMB process

PDE system (couple N_{Col} columns)



$$\frac{\partial C_i^n(t,z)}{\partial t} + \frac{1-\varepsilon}{\varepsilon}\frac{\partial q_i^n(t,z)}{\partial t} = -u_n \frac{\partial C_i^n(t,z)}{\partial z} + D_n \frac{\partial^2 C_i^n(t,z)}{\partial z^2}, \quad i = A,B \quad n = 1,2,\ldots,N_{Col}$$

$$\frac{\partial q_i^n(t,z)}{\partial t} = Km_i(q_i^{n,Eq}(t,z) - q_i^n(t,z)), \quad i = A,B$$

$$q_i^{n,Eq} = f_i(C_A^n, C_B^n) = \frac{H_{i,1}C_i^n}{1+K_{A,1}C_A^n + K_{B,1}C_B^n} + \frac{H_{i,2}C_i^n}{1+K_{A,2}C_A^n + K_{B,2}C_B^n}$$

Initial and boundary conditions:

$$C_i^n(t=0,z) = 0, \quad q_i^n(t=0,z) = 0$$

$$\left.\frac{\partial C_i^n}{\partial z}\right|_{z=0} = \frac{u_n}{D_n}(C_i^n(t,0) - C_i^{n,in}(t)), \qquad \left.\frac{\partial C_i^n}{\partial z}\right|_{z=L} = 0$$

Motivation

Mathematical modeling of SMB process

PDE system (couple N_{Col} columns) —spatial discretization→ **DAE system**

A complex system of nonlinear, parametric DAEs:

$$M(\mu)\frac{dx}{dt} = A(\mu, x) + B,$$
$$y(t) = Cx(t),$$

μ : operating conditions

with proper initial conditions.

Basic Idea of MOR

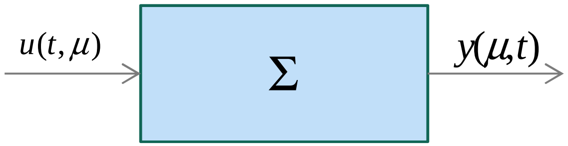
Original model (discretized)

$$\Sigma \begin{cases} M(\mu)\dfrac{dx}{dt} = A(\mu, x) + B, \\ y(t) = Cx(t), \end{cases}$$

states $x \in \mathrm{R}^{n}$,

inputs $u(t) \in \mathrm{R}^{m}$

output $y(t) \in \mathrm{R}^{q}$



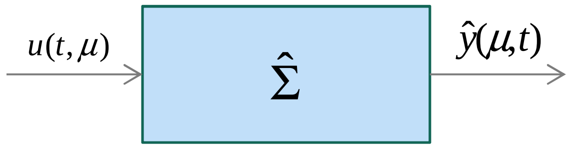
Reduced order model

$$\hat{\Sigma} \begin{cases} \hat{M}(\mu)\dfrac{dz}{dt} = \hat{A}(\mu, z) + \hat{B}, \\ \hat{y}(t) = \hat{C}z(t), \end{cases}$$

states $\hat{x} \in \mathrm{R}^{r}$,

inputs $u(t) \in \mathrm{R}^{m}$

output $\hat{y}(t) \in \mathrm{R}^{q}$



Goal: $\| y - \hat{y} \| < \mathrm{tol} \quad \forall u(t, \mu)$

Basic Idea of MOR

Original model → (MOR) → Reduce order model (ROM)

↓

Replace the original model with the reduced model. The reduced model is then integrated with other components in the device or circuit; or is integrated into a process.

↓

The reduced model is repeatedly used in design analysis.

Basic Idea of MOR

Example: Optimization for SMB

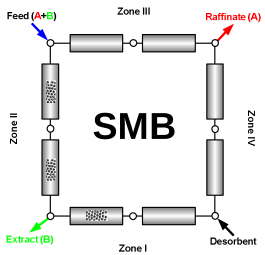
$$\max_{p \in R^5} \; Q_F$$

s.t.

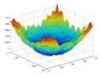
- *SMB model (PDEs)*
- *Cyclic steady state (CSS) constraints*
- *Product purity constraints:* $Pur_A \geq Pur_{A,min}, \; Pur_B \geq Pur_{B,min}$
- *Operational constraints on p*

Multi-query, real-time simulation?

ROMs ⇒

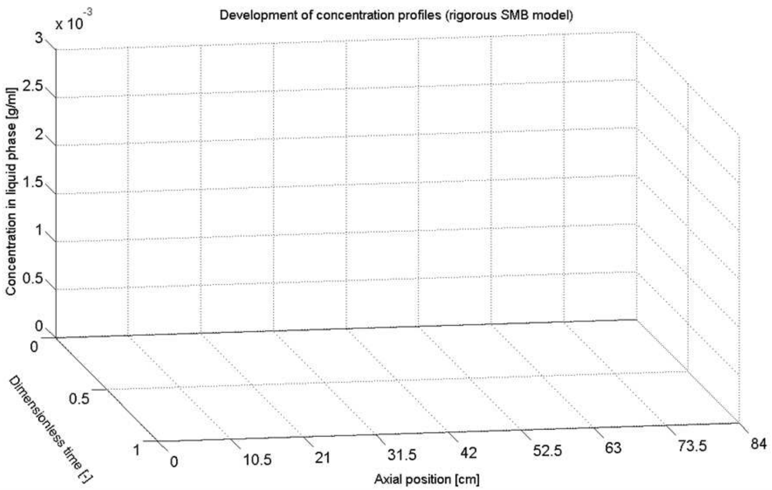


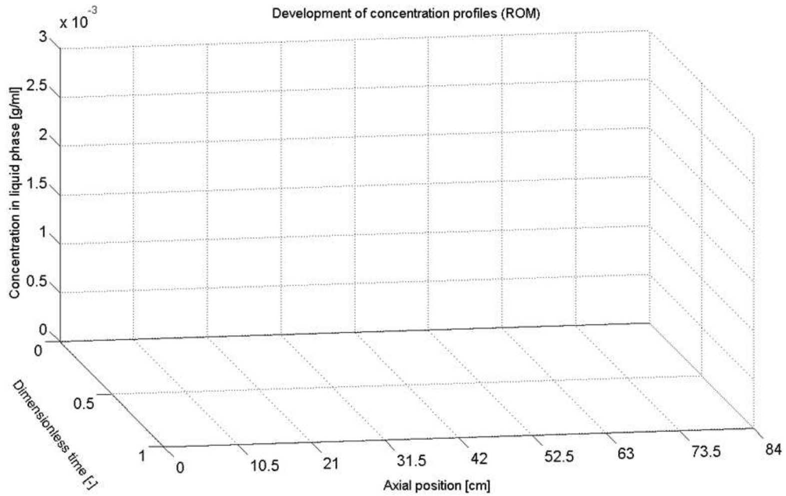
⇒



ROM-based optimization

Basic Idea of MOR



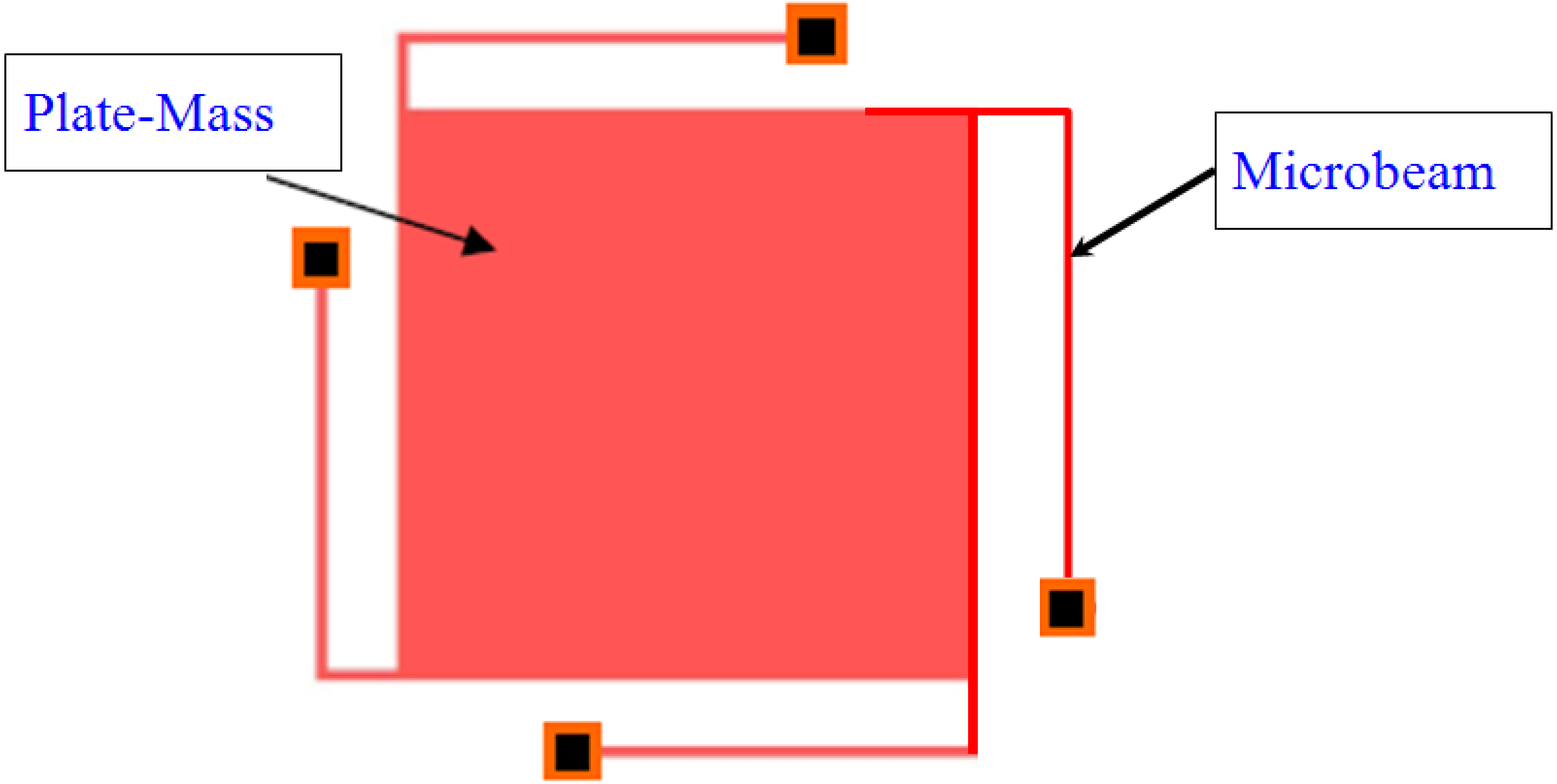


Axial concentration profiles of the full-order DAE model with order of 672.

Axial concentration profiles reproduced by POD-based ROM (with reduced order of only 2).

Basic Idea of MOR

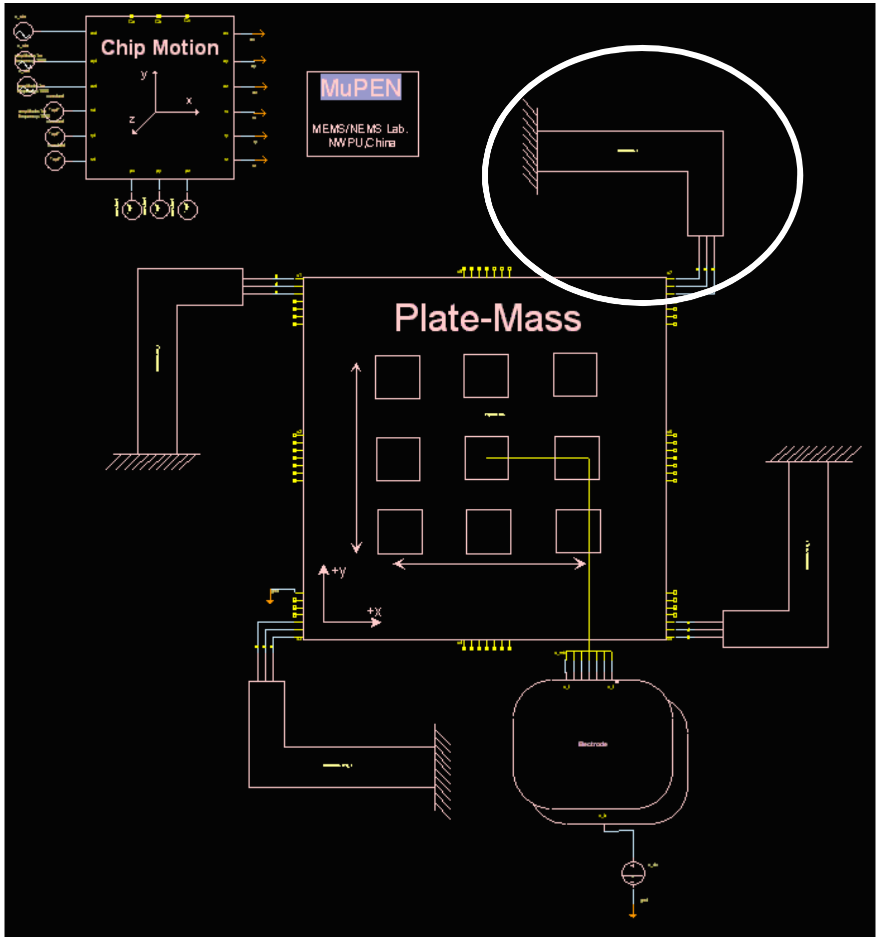
Example: Layout of a switch with four microbeams

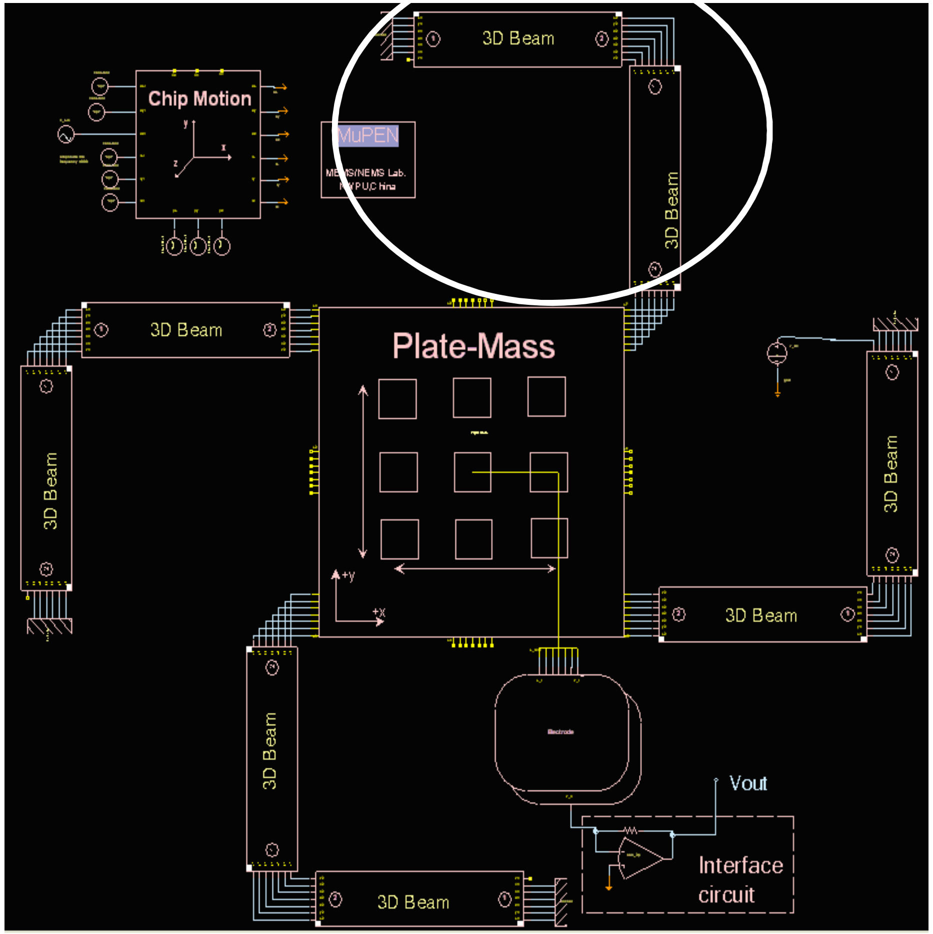


Basic Idea of MOR

The microbeam is replaced by the ROM

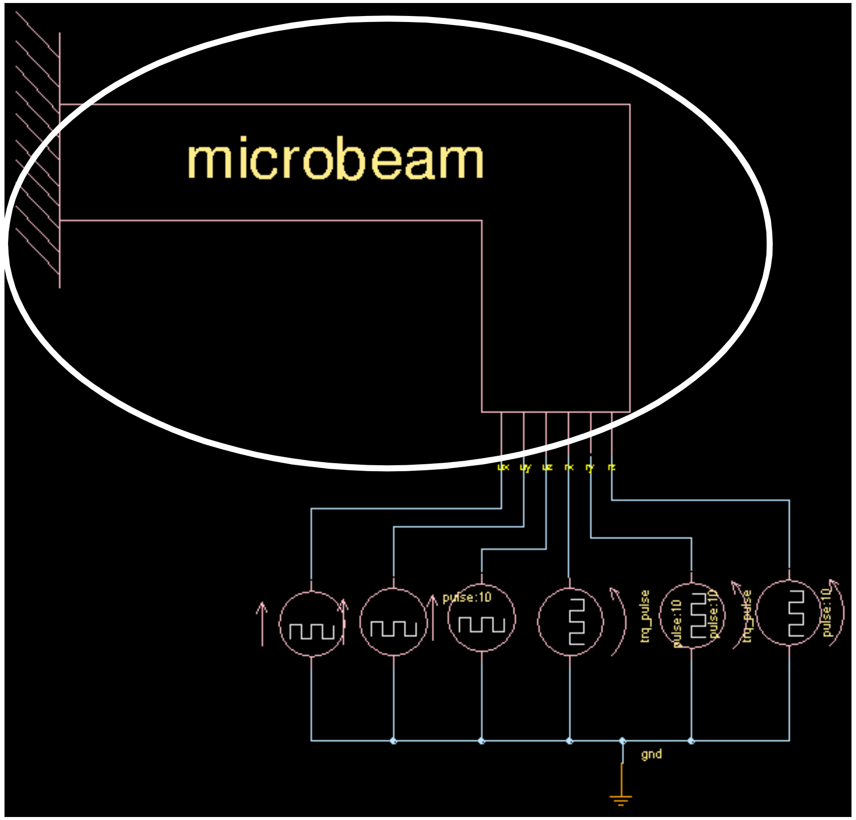
The schematic switch

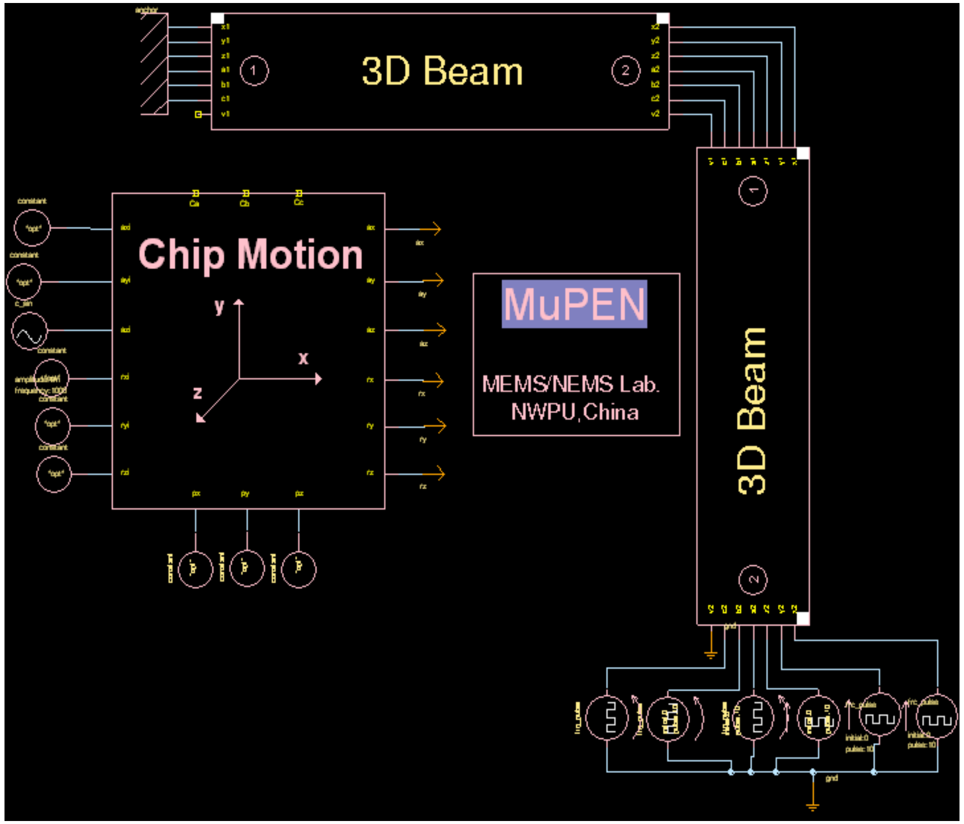




ROM of the microbeam

Enlarged microbeam Part





Projection based MOR

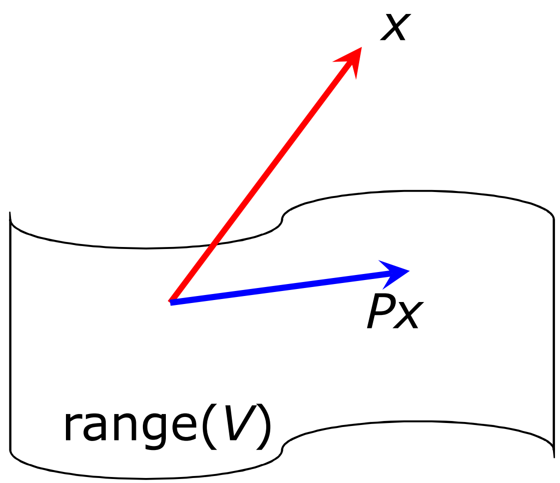
Original model (discretized)

$$\Sigma \begin{cases} M(\mu)\dfrac{dx}{dt} = A(\mu, x) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$

Reduced order model

$$\hat{\Sigma} \begin{cases} \hat{M}(\mu)\dfrac{dz}{dt} = \hat{A}(\mu, z) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}z(t), \end{cases}$$

Find a subspace which includes the trajectory of *x*, use the projection of x in the subspace to approximate x.



Let: $x \approx Vz$

$$M(\mu)V\frac{dz}{dt} = A(\mu, Vz) + Bu(t) + e,$$

$$\hat{y}(t) = CVz(t),$$

Petrov-Galerkin projection: $W^T e = 0$, $\text{range}(W) = \text{span}\{w_1, \ldots, w_r\}$.

$e = 0$ in $\text{range}(W) \Leftrightarrow w_i^T e = 0$ for all $i = 1, \ldots, n$.

$$M(\mu)V\frac{dz}{dt} = A(\mu, Vz) + Bu(t) + e,$$
$$\hat{y}(t) = CVz(t),$$

$\Updownarrow$

$$W^T M(\mu)V\frac{dz}{dt} = W^T A(\mu, Vz) + W^T Bu(t),$$
$$\hat{y}(t) = CVz(t),$$

$$\hat{M} = W^T MV, \hat{A} = W^T A(\mu, Vz), \hat{B} = W^T B, \hat{C} = CV.$$

Conclusions

The question:

How to construct the ROMs (W, V) for the large-scale complex systems?

Answer:

The lecture will provide many solutions.

Outline of the Lecture

- Mathematical basics
- Moment-matching method for linear time invariant systems.
- Balanced truncation method for linear time invariant systems.
- Krylov subspace based method for nonlinear systems and POD method for nonlinear systems.
- Krylov subspace based method for parametric system.

- **Notice: time and location changes**
- Lectures on June 12, and June 26 fall out.
- There will be Lectures on June 17 and June 30, 09:00-11:00, in Room Seminarv0.05 2+3 in MPI.

- http://www.mpi-magdeburg.mpg.de/2492919/mor_ss14