Tutorial

Intrinsic Low–Dimensional Manifolds and Slow Attractors

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Abstract

Section 1 presents the algorithm for the computation of the *intrinsic low–dimensional manifold* of Maas & Pope for systems of ordinary differential equations with expected slow and fast dynamics (from [4] or [5]). Section 2 offers an example where this algorithm produces an intrinsic manifold which is neither (approximately) invariant nor (approximately) slow. Section 3 presents an example where the intrinsic manifold is reasonably close to a slow invariant manifold but where the reduced equations of Maas & Pope are not suited to predict the true dynamical behavior of the full system. The remaining sections 4 and 5 are dedicated to the attractivity of the intrinsic manifold. It is shown that the Maas & Pope algorithm can lead to erroneous statements about the attractivity of the intrinsic manifold – even locally near (unstable) equilibria. Moreover these sections reveal that the dimension of the intrinsic manifold of Maas & Pope need not coincide with the dimension of the (true) slow and attracting invariant manifold of the underlying system.

Contents

1 Intrinsic Manifolds by Maas & Pope

We give a short informal outline how Maas & Pope derive their intrinsic slow manifolds in [4] or [5] and assume enough smoothness for the derivations to hold on appropriate domains. Given an ODE

$$\dot{\xi} = F(\xi) \tag{1.1}$$

in $\mathbb{R}^N$ with Jacobian

$$Jac(\xi) \equiv DF(\xi) \tag{1.2}$$

Maas & Pope assume to have a block–diagonalizing matrix $Q(\xi)$ with

$$Jac(\xi)Q(\xi) = Q(\xi)\begin{pmatrix} A(\xi) & 0 \\ 0 & B(\xi) \end{pmatrix} \tag{1.3}$$

where the eigenvalues of $B(\xi)$ are to the left of those of $A(\xi)$. Let us formulate this with the help of a positive ρ having the property

$$Re\ \sigma(B(\xi)) < -\rho < Re\ \sigma(A(\xi))\ . \tag{1.4}$$

In addition Maas & Pope assume $Re\ \sigma(A(\xi))$ to be near to 0. One may then speak of the eigenvalues $\lambda_s(\xi)$ of $A(\xi)$ as the *slow frozen eigenvalues* in contrast to the *fast frozen eigenvalues* $\lambda_f(\xi)$ of $B(\xi)$.

!!! Recall how the smoothness of center manifolds of an equilibrium E^* depends on the gap between the realpart of the spectrum of $A(E^*)$ and $B(E^*)$ and its position. Similarily for singular perturbation problems in standard form as in [2]. So one should probably ask for $Re\ \sigma(A(\xi))$ to be *arbitrarily* close to 0 or for $Re\ \sigma(B(\xi))$ to be *arbitrarily* far left.

By partitioning $Q(\xi)$ acoording to the dimensions of $A \in A^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ and by writing $\xi = (x, y)^{\mathrm{T}}, F = (f, g)^{\mathrm{T}}$ Maas & Pope define their *intrinsic low–dimensional slow manifold* for (1.1) by those ξ for which the vector field $F(\xi)$ lies in the *slow frozen eigenspace*, i.e. by

$$F(x, y) = Q(x, y)\begin{pmatrix} \alpha(x, y) \\ 0 \end{pmatrix}. \tag{1.5}$$

This system is equivalent to the *reduction equations* given by

$$(0\ I^{m\times m})Q^{-1}(x, y)F(x, y) = 0. \tag{1.6}$$

The solution $y = s(x)$ (when- and wherever it exists) is then the so–called *intrinsic low–dimensional slow manifold*. The (slow?) reduced dynamics according to [4] are then given by

$$\dot{x} = f(x, s(x)). \tag{1.7}$$

- One should ask the question why this purely algebraic reduction process should respect the underlying dynamical structure of (1.1). In general, $y = s(x)$ is not invariant (cf. the examples (1.15) or (1.21)).

We mention the special case where (1.1) is in the form

$$\dot{x} = f(x), \quad \dot{y} = g(x, y) \tag{1.8}$$

where the block–diagonalizing $Q(x, y)$ can be taken of the form

$$Q(x, y) = \begin{pmatrix} I & 0 \\ Z(x, y) & I \end{pmatrix} \quad \text{with} \ \ g_x + g_y Z = Z f_x \ . \tag{1.9}$$

For scalar x and y one just has

$$Z(x, y) = -[g_y(x, y) - f_x(x)]^{-1} g_x(x, y). \tag{1.10}$$

The defining equation (1.5) or (1.6) then reads

$$0 \overset{!}{=} (-Z \ I) \begin{pmatrix} f \\ g \end{pmatrix} = g - Zf \tag{1.11}$$

and its solution $y = s(x)$ would be the intrinsic slow manifold of Mass & Pope for (1.8). Its derivative $s_x(x)$ – if existent – can be obtained by implicit differentiation of

$$g(x, s(x)) - Z(x, s(x)) f(x) \equiv 0 \ . \tag{1.12}$$

In contrast, an invariant manifold $y = \sigma(x)$ of (1.8) necessarily satisfies the PDE

$$0 = \dot{y} - \sigma_x(x)\dot{x} = g(x, \sigma(x)) - \sigma_x(x) f(x) \tag{1.13}$$

respecting the underlying dynamics.

- In general there is no reason for $y = s(x)$ from (1.12) to satisfy (1.13) – not even approximately. So there is no reason to believe that there is a genuine invariant manifold of (1.8) close to $y = s(x)$.
- In general there is no reason for the dynamics near $y = s(x)$ to be slow since the conditions (1.6) or (1.11) impose restrictions only on the direction of the vector field and not on its size.

In case $\dot{x} = 0$ or in case $\dot{x} = \varepsilon f(x, y)$ the method of [4] leads to an intrinsic manifold $y = s(x, \varepsilon)$ with $g(x, s(x, 0)) \equiv 0$ so that the intrinsic manifold of Maas and Pope is close to the quasi–static manifold $y = \sigma_0(x)$ of singular perturbation theory for

$$\dot{x} = \varepsilon f(x, y, \varepsilon), \quad \dot{y} = g(x, y, \varepsilon) \tag{1.14}$$

with $g(x, \sigma_0(x), 0) \equiv 0$ and Hurwitz $g_y(x, \sigma_0(x), 0)$ (cf. [2]). See e.g. the discussion of (1.21) or (1.24).

A simple example:

Let's consider the following situation

$$\dot{x} = \varepsilon f(x), \quad \dot{y} = g(x) - By \tag{1.15}$$

with the Jacobian

$$Jac(x) = \begin{pmatrix} \varepsilon f'(x) & 0 \\ g'(x) & -B \end{pmatrix} \tag{1.16}$$

and a Hurwitz matrix $-B$. Choose $Z = Z(x, \varepsilon)$ with

$$J \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \begin{pmatrix} \varepsilon f'(x) & 0 \\ 0 & -B \end{pmatrix}, \quad i.e.\ \ g'(x) - BZ = Z\varepsilon f'(x) \tag{1.17}$$

Condition (1.11) forces the vector field (1.15) into the frozen eigenspaces via

$$0 \stackrel{!}{=} (-Z, I) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = g(x) - By - Z\varepsilon f(x). \tag{1.18}$$

The intrinsic manifold is then the solution of the reduction equation (1.18) and thus

$$y = s(x, \varepsilon) = B^{-1}[g(x) - \varepsilon Z(x, \varepsilon) f(x)] \tag{1.19}$$

with

$$\begin{gathered} \dot{y} - s_x(x, \varepsilon)\dot{x} = \\ g(x) - [g(x) - \varepsilon Z(x,\varepsilon) f(x)] - B^{-1}[g'(x) - \varepsilon Z(x,\varepsilon) f'(x) - \varepsilon [Z(x,\varepsilon) f'(x)]']\varepsilon f(x) = \\ \varepsilon^2 B^{-1}[Z(x,\varepsilon) f'(x)]' f(x) \end{gathered} \tag{1.20}$$

by (1.18). Thus:

Condition (1.11) does in general not define an invariant manifold.

This can be seen by the 2D–example

$$\dot{x} = \varepsilon x, \quad \dot{y} = g(x) - y. \tag{1.21}$$

with a nonlinear $g(x)$. As soon as $g''(x)$ does not vanish for all x the intrinsic manifold

$$y = s(x, \varepsilon) \equiv g(x) - \tfrac{\varepsilon}{1+\varepsilon} x g'(x) = g(x) - \varepsilon x g'(x) + \cdots \tag{1.22}$$

given by (1.11) is not invariant since one has

$$\dot{y} - s_x(x, \varepsilon)\dot{x} = \tfrac{\varepsilon^2}{1+\varepsilon} x^2 g''(x). \tag{1.23}$$

In general, for 2–dimensional singular perturbation problems

$$\dot{x} = \varepsilon f(x, y), \quad \dot{y} = g(x, y) \tag{1.24}$$

with a nondegenerate zero $y = s_0(x)$ of g (satisfying $g_y(x, s_0(x)) \overset{!}{<} 0$) the condition (1.11) gives the first order approximation $y = s_0(x) + \varepsilon s_1(x)$ of the true slow manifold of (1.24).

Remark:

The system (1.21) with $g(x) = x^2$ and $\varepsilon \in (0, 1)$, i.e. the system

$$\dot{x} = \varepsilon x, \quad \dot{y} = x^2 - y \tag{1.25}$$

shows that the intrinsic manifold

$$y = s(x, \varepsilon) = \frac{1 - \varepsilon}{1 + \varepsilon} x^2$$

is off the slow manifold of (1.25) which is given by the unstable manifold

$$y = \sigma(x, \varepsilon) = \frac{1}{1 + 2\varepsilon} x^2$$

of the equilibrium $(0, 0)$. For negative ε it is not so easy to define what the(?) slow manifold should be.

- This example also illustrates that the approach of Maas and Pope is dependent on the chosen coordinates: In the new coordinates

  $$u = x, \quad v - y - \sigma(x, \varepsilon)$$

  leading to $\dot{u} = \varepsilon u,\ \dot{v} = -v$ one has the intrinsic manifold $v \equiv 0$ and not $v = s(x.\varepsilon) - \sigma(x, \varepsilon)$.

- Furthermore we note that the first two terms in the expansions of s and σ coincide (as it is the case in general – cf. [3]).

For simulations of (1.25) and of

$$\dot{x} = \varepsilon x(x - 1)(x + 2), \quad \dot{y} = x^2 - y \tag{1.26}$$

with $\varepsilon \in (-\frac{1}{6}, \frac{3}{7})$ over $-2 \le x \le 1$ see the Figures 1 – 6.

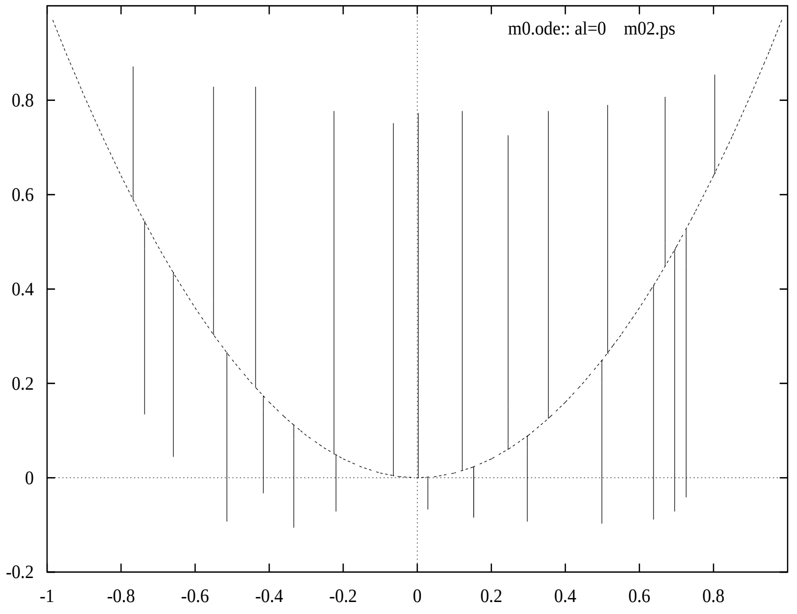

Figure 1: m02.ps

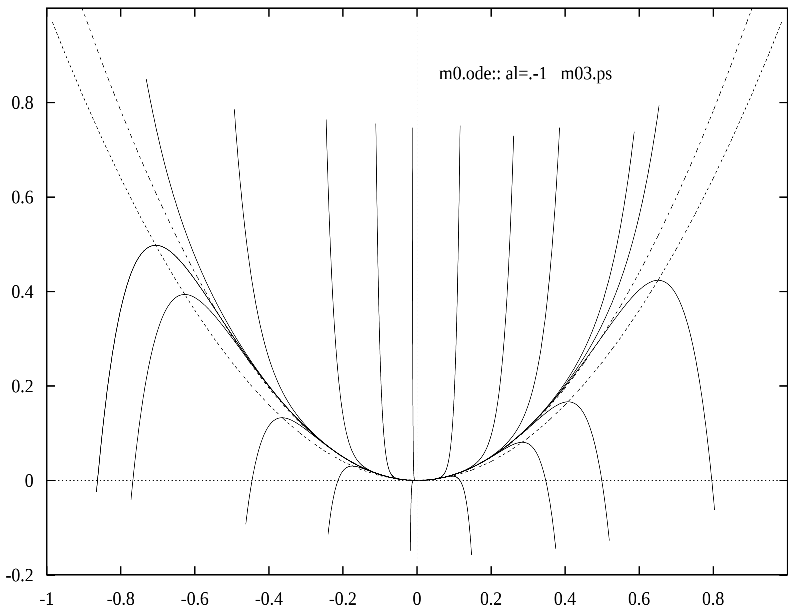

Figure 2: m03.ps

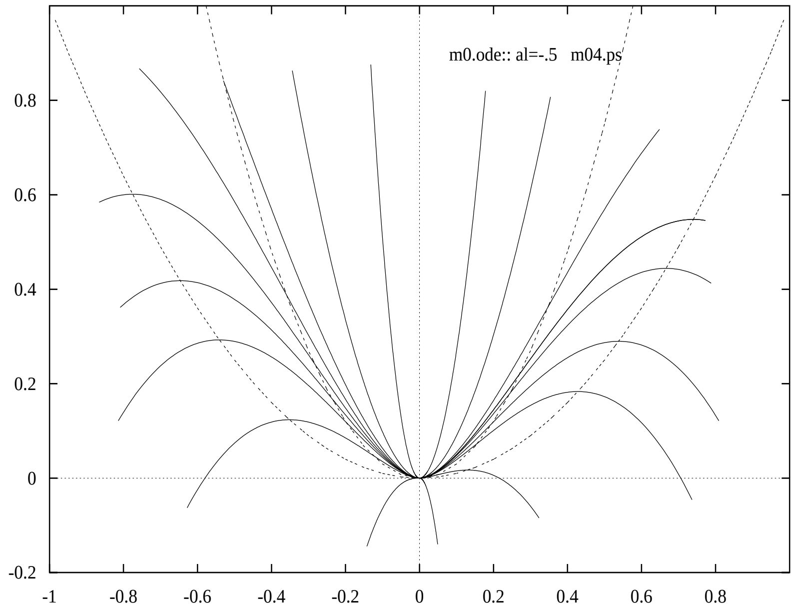

Figure 3: m04.ps

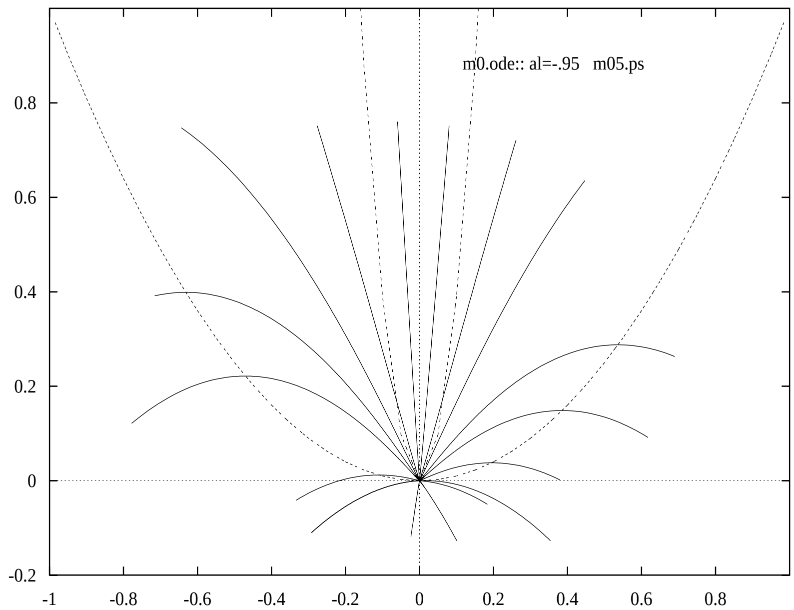

Figure 4: m05.ps

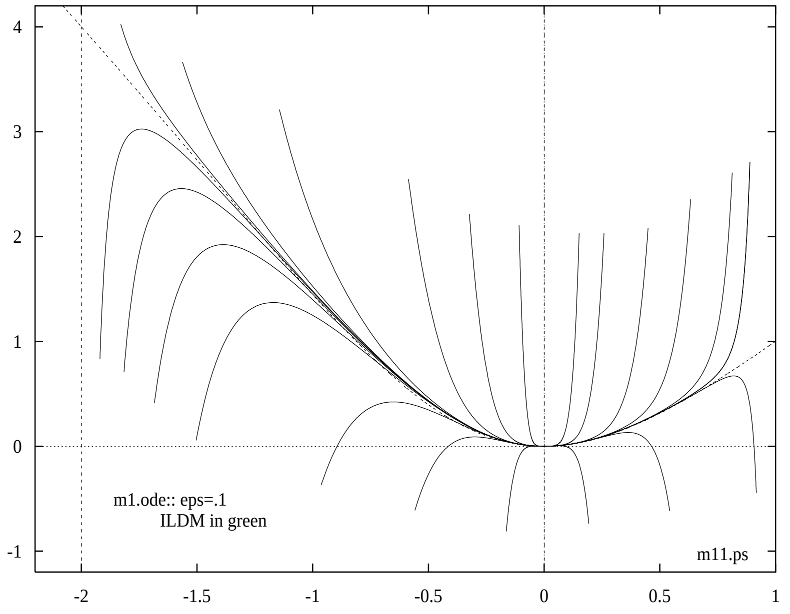

Figure 5: m11.ps, dotted ILDM

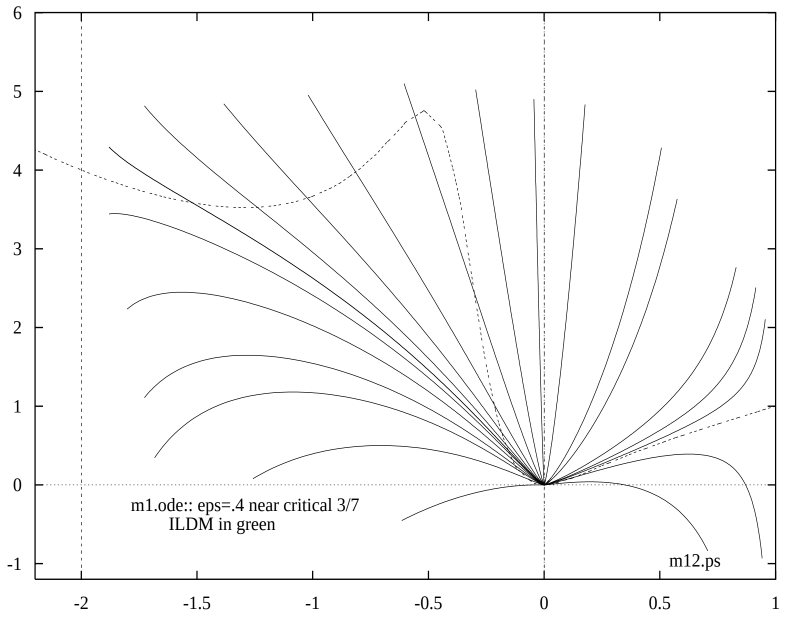

Figure 6: m12.ps, dotted ILDM

2 Ghost Manifold

We present an $2D$-example where the method Maas & Pope produces a ghost slow manifold, i.e. a curve $y = s(x)$ which is not close to a genuine invariant manifold and where the dynamics are not slow at all. Consider the system

$$\begin{aligned} \dot{x} &= f(x) = 3 - \varepsilon(x + 4.5), \\ \dot{y} &= g(x, y) = -(x + 0.5)(\beta y - g_0) - \gamma(x + 4.5)^2 \end{aligned} \tag{2.1}$$

with small ε, γ ($\varepsilon = \gamma = .1$ e.g.) and $\beta > 2\varepsilon$ ($\beta = 2$ e.g.) in the right half–plane so that $f_x = -\varepsilon > \frac{\beta}{2} \geq = -(x + 0.5)\beta = g_y$. The constant $g_0 = 40.5\gamma$ is chosen to have $g(0, 0) = 0$. The block–diagonalizing Q of (1.9) is given by (1.10) which now reads

$$Z(x, y) = [-\beta(x + 0.5) + \varepsilon]^{-1}[\beta y - g_0 + 2\gamma(x + 4.5)] \tag{2.2}$$

for $x \geq 0$. The condition (1.11) for the computation of the intrinsic manifold is thus

$$0 \stackrel{!}{=} [-\beta(x + 0.5) + \varepsilon]g(x, y) - [\beta y - g_0 + 2\gamma(x + 4.5)]f(x). \tag{2.3}$$

It is linear in y with two solution branches on $x \geq 0$ given by

$$\beta y - g_0 = \gamma(x + 4.5)\ \frac{2\cdot 3 - \beta(x+0.5)(x+4.5) - \varepsilon(x+4.5)}{\beta(x+0.5)^2 - (3-4\varepsilon)} \equiv \gamma Y(x) \tag{2.4}$$

with a simple pole at the positive zero

$$x_+ = -0.5 + \sqrt{(3 - 4\varepsilon)/\beta} \quad (x_+ = .64... \text{ for } \varepsilon = .1, \beta = 2) \tag{2.5}$$

of the denominator for $\beta < 12 - 16\varepsilon$. The intrinsic manifold has the branches $y = s_L(x)$ and $y = s_R(x)$ given by the same formula

$$y = \tfrac{\gamma}{\beta}[40.5 + Y(x)] \tag{2.6}$$

over $[0, x_+)$ and (x_+, ∞) respectively. At the left branch the dynamics are transversal and not slow at all – all one has is that the vectorfield points into the direction of the slow frozen eigenspace spanned by $(1, Z(x, s_L(x)))$. The reduced dynamics are erroneous. In contrast, the right branch is near a genuine 'slow' invariant manifold with the equilibrium

$$E^* = (x^*, y^*) = (-4.5 + \frac{3}{\varepsilon}, \frac{1}{\beta}[g_0 - \frac{9\gamma}{\varepsilon(3 - 4\varepsilon)}])$$

being ω–limit set for any initial in the right half–plane. The 'slow' invariant manifold is given by the(?) slow stable manifold of E^* corresponding to the eigenvalue $-\varepsilon$ – the fast stable manifold of E^* is associated with the eigenvalue $-(3 - 4\varepsilon)\frac{\beta}{\varepsilon}$. For simulations of (2.1) see Figure 7.

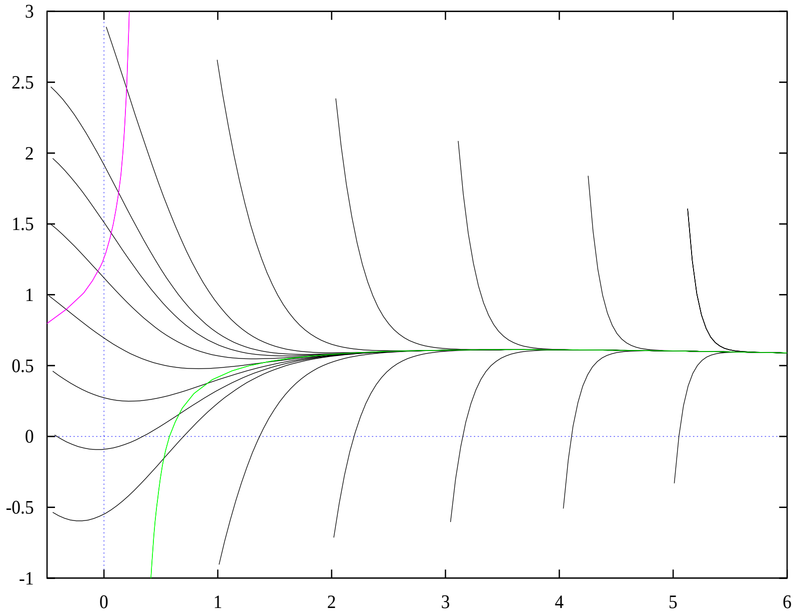

Figure 7: Erroneous ILDM-branch in magenta (C2-fig.eps)

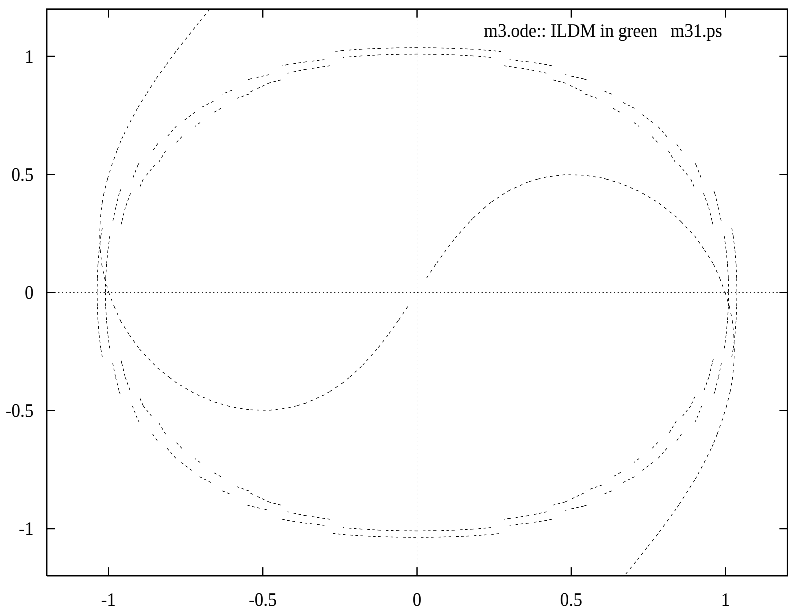

Figure 8: m31.ps

3 Ghost Dynamics

We present an $2D$-example where the method Maas & Pope produces an approximate slow manifold close to a genuine slow invariant manifold where the reduced dynamics do not predict the correct dynamical behavior. Consider the system

$$\dot{x} = f(x, y) \equiv (1 - r^2)x - \varepsilon y, \quad \dot{y} = g(x, y) \equiv \varepsilon x + (1 - r^2)y \tag{3.1}$$

which has the polar representation

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = \varepsilon \tag{3.2}$$

showing the slow exponentially attracting invariant circle $\mathcal{C}_0 = \{r = 1, 0 \leq \theta < 2\pi\}$ for (3.1). A lengthy calculation shows that the intrinsic manifold of Maas & Pope is given by

$$\mathcal{C}_\varepsilon = \{r^4 = 1 + \varepsilon^2, 0 \leq \theta < 2\pi\}. \tag{3.3}$$

On it the reduced dynamics (see section 6)

$$\dot{x} = -(1 - \sqrt{1 + \varepsilon^2})x \pm \varepsilon\sqrt{1 + \varepsilon^2 - x^2} \tag{3.4}$$

have stable equilibria near $x = 1$ for the lower and $x = -1$ for the upper open half–circle for small $\varepsilon > 0$. The points $(x, y) = (\pm[1 + \varepsilon^2]^{1/4}, 0)$ are critical in these charts – one should switch to the left and right open half–circle there (with analogous implications). For simulations of (3.1) see Figure 8.

4 Ghost Attractor I

Consider the 3–dimensional ODE

$$\dot{x} = 1, \quad \dot{y} = B(x)y \tag{4.1}$$

with initial value $x(0) = 0$ for the scalar x so that $x(t) \equiv t$ follows. The matrix $B(x)$ is defined via

$$B(x) = e^{-Jx}B_0e^{Jx} \quad \text{with } B_0 = \begin{pmatrix} -\beta_1 & -\gamma \\ 0 & -\beta_2 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.2}$$

where β_1, β_2 and

$$\gamma \geq 1 + \beta_1\beta_2 \tag{4.3}$$

are positive parameters. By similarity, $B(x)$ has the *frozen eigenvalues* $-\beta_1$ and $-\beta_2$. The regular (3×3)–matrix

$$Q(x,y) = \begin{pmatrix} 1 & 0 \\ -\Gamma(x)y & I \end{pmatrix} \quad \text{with} \;\; \Gamma(x) = B^{-1}(x)B'(x) \tag{4.4}$$

block–diagonalizes the frozen Jacobian of (4.1) since one has

$$\begin{pmatrix} 1 & 0 \\ \Gamma(x)y & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B'(x)y & B(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Gamma(x)y & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B(x) \end{pmatrix}. \tag{4.5}$$

By (1.6) the *intrinsic low–dimensional manifold* is now given by

$$(\Gamma(x)y \;\; I) \begin{pmatrix} 1 \\ B(x)y \end{pmatrix} = [\Gamma(x) + B(x)]y \stackrel{!}{=} 0. \tag{4.6}$$

The only solution of (4.6) is $y \equiv 0$ since $\Gamma(x) + B(x)$ is regular for all x.

- Proof: One has $\Gamma + B = B^{-1}[B' + B^2] = B^{-1}e^{-Jx}Ce^{Jx}$ with

$$C \equiv -JB_0 + B_0J + B_0^2 = \begin{pmatrix} \gamma + \beta_1^2 & \beta_2 - \beta_1 + (\beta_1 + \beta_2)\gamma \\ \beta_2 - \beta_1 & -\gamma + \beta_2^2 \end{pmatrix}$$

  with regular matrices B, e^{Jx} and C – recall (4.3).

The intrinsic manifold is thus the 1–dimensional x–axis

$$M_1 = \{(x,y) : \; x \in \mathbb{R} \;, y = 0\}. \tag{4.7}$$

It is supposed to be an approximation to the slow attractor for (4.1) (according to [4]). This approximation is in general not invariant, in the present case it is. But M_1 is *not* exponentially attractive as the following computation of the Floquet–exponents will show. Let's call M_1 a ghost attractor.

The transformation

$$z = e^{Jt}y \tag{4.8}$$

transforms (4.1) into

$$\dot{z} = [J + B_0]z = \begin{pmatrix} -\beta_1 & 1 - \gamma \\ -1 & -\beta_2 \end{pmatrix} z \tag{4.9}$$

with the eigenvalues

$$\mu_{1/2} = -\tfrac{1}{2}[\beta_1 + \beta_2] \pm \sqrt{\tfrac{1}{4}[\beta_1 + \beta_2]^2 - (\beta_1\beta_2 - \gamma + 1)}. \tag{4.10}$$

These eigenvalues are the Floquet–exponents of (4.1) with $\mu_1 \geq 0$ and $\mu_2 < 0$. Let W_1, W_2 be corresponding eigenvectors for (4.9). For $\gamma = 1 + \beta_1\beta_2$ the slow attractor is the 2–dimensional

$$M_2 = \{(x,y) : \; x \in \mathbb{R}, y = e^{-Jx}W_1\} \tag{4.11}$$

carrying the solutions

$$x(t) = t,\ y(t) = e^{-Jt}W_1\xi, \quad \xi \in \mathbb{R} \tag{4.12}$$

of vanishing exponential growth rate. M_2's stable manifold is generated by the rotation e^{-Jx} of W_2.

M_1 is for $\gamma > 1 + \beta_1\beta_2$ of saddle type with 1–dimensional unstable and 1–dimensional stable manifold (generated by rotated W_1 and W_2 respectively).

If the β_j tend to ∞ and if $\gamma = \mathcal{O}(\beta_1)$ the above example has for sufficiently large β_j indeed M_1 as slow attractor. It serves as a counterexample only for $\gamma \geq 1 + \beta_1\beta_2$.

Remark:

- An interesting (amazing?) modification of (4.1) is

$$\dot{x} = \alpha, \quad \dot{y} = B(x)y \tag{4.13}$$

  with positive α, $\beta_1 = \beta_2$ and $\gamma = 1 + \beta^2$. The transformation (4.9) then leads to

$$\dot{z} = [\alpha J + B_0]z = \begin{pmatrix} -\beta & \alpha - \gamma \\ -\alpha & -\beta \end{pmatrix} z \tag{4.14}$$

  instead of (4.10). The determinant of $\alpha J + B_0$ is $(\beta^2 - \alpha)(1 - \alpha)$. The eigenvalues of (4.14) are

$$\mu_{1/2} = -\beta \pm \sqrt{\alpha(1 + \beta^2 - \alpha)}. \tag{4.15}$$

  One thus obtains 2–dimensional slow attractors for $\alpha = 1$ and for $\alpha = \beta^2$ and 1–dimensional ones for

$$\alpha < min(1, \beta^2) \quad \text{and} \quad \alpha > min(1, \beta^2). \tag{4.16}$$

  1–dimensional slow invariant manifolds of saddle–type exist for

$$1 < \alpha < \beta^2 \quad \text{and} \quad 1 > \alpha > \beta^2. \tag{4.17}$$

  The problem in a singular perturbation context is the one with fixed β and $\alpha \to 0$ (first case in (4.16)). Here, the frozen eigenvalues determine the exponential attractivity of $\{y = 0\}$ for sufficiently small α.

- Similarily, consider (4.13) with $\alpha = \beta_1 = \beta_2 =: \beta$ and $\gamma = 2\beta$ which can be written as

$$\dot{x} = \beta, \quad \tfrac{1}{\beta}\dot{y} = B^*(x)y \equiv e^{-Jx} \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} e^{Jx} y \tag{4.18}$$

  or in scaled time as

$$\dot{x} = 1, \quad \dot{y} = B^*(x)y. \tag{4.19}$$

  The x–dynamic is non–exponential and hence slow, the y–dynamics are determined by the Floquet–exponents 0 and -2. The slow attractor is thus $2D$–dimensional in contrast to the first impression one may obtain by looking at the y–equation in (4.18).

For simulations of (4.1) see Figure 9.

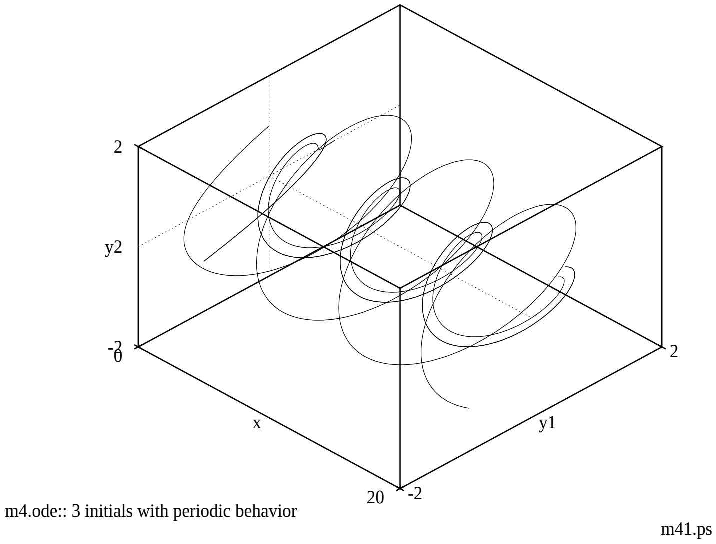

Figure 9: m41.ps

5 Ghost Attractor II

We present a modification of the example of section 4 where the relevant portion of the phase space is compact and where there exists an equilibrium for the underlying system. Consider

$$\begin{aligned} \dot{r} &= R(r) = r(r^2 - \tfrac{1}{4})(1 - r^2), \\ \dot{\theta} &= \Theta(r, \theta) = 1 - \tfrac{8}{3}(1 + p)(1 - r^2) r sin(\theta), \\ \dot{y} &= B(\theta) y \end{aligned} \tag{5.1}$$

with $p \geq 0$ and with $B(\theta)$ from (4.2). For simplicity of the presentation we assume

$$\beta_1 = \beta_2 \equiv \beta, \quad \gamma = 1 + \beta^2. \tag{5.2}$$

We restrict the phase space in polar coordinates to

$$\tfrac{1}{2} \leq r \leq 1, \quad 0 \leq \theta < 2\pi. \tag{5.3}$$

On the circle $r = 1$ one has the example of section 4, on the circle $r = \frac{1}{2}$ one has a saddle node $(r, \theta) = (\frac{1}{2}, \frac{\pi}{2})$ for $p = 0$ splitting into a saddle $(\frac{1}{2}, \theta_1)$ and an unstable node $(\frac{1}{2}, \theta_2)$ with $\theta_1 < \frac{\pi}{2} < \theta_2$ for small $p > 0$. In the (r, θ)–plane with $\frac{1}{2} < r \leq 1$ the ω–limit set of any trajectory is the circle $r = 1$. Hence one has:

- The circle $r = \frac{1}{2}, y = 0$ is a $1D$ slow manifold with a $2D$ fast exponentially stable manifold of exponential rate close to $-\beta t$ for $t \to \infty$.
- Over the annulus $\frac{1}{2} < r \leq 1, y = 0$ one just has one fast direction towards a $3D$ slow manifold (cf. section 4). The annulus itself is part of the $3D$ slow manifold. The slow attractor is the restriction to $r = 1$.

Even in this geometrically simple example it is rather tedious to compute the intrinsic manifold of Maas & Pope. It turns out to be the $2D$ annulus $\frac{1}{2} \leq r \leq 1$. We just present the essentials of this derivation.

The frozen Jacobian of (5.1) is of the form

$$Jac = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \tag{5.4}$$

with

$$A = \begin{pmatrix} R_r & 0 \\ \Theta_r & \Theta_\theta \end{pmatrix}, \quad C = \begin{pmatrix} 0 & C_{12} \\ 0 & C_{22} \end{pmatrix}, \quad \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix} = e^{-J\theta} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} e^{J\theta}\, y\ . \tag{5.5}$$

Thereby one has

$$R_r \in [-\tfrac{3}{2}, \tfrac{3}{8}], \quad |\Theta_\theta| \leq \tfrac{16}{27}\sqrt{3} \sim 1.0264...\ . \tag{5.6}$$

For β bigger than $\frac{3}{2}$ the frozen eigenvalues of J are splitted into the slow ones of A and the fast stable ones of B as asked for in (1.4). The slow frozen eigenspaces possess a basis of the form

$$\begin{pmatrix} I \\ Z \end{pmatrix}$$

with

$$\begin{aligned} Z(r,\theta,y) &= e^{-J\theta}(U(r,\theta)y, V(r,\theta)y), \\ U(r,\theta) &= -\Theta_r(R_r - B_0)^{-1}V(r,\theta), \\ V(r,\theta) &= (\Theta_\theta - B_0)^{-1} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} e^{J\theta}\ . \end{aligned} \tag{5.7}$$

The defining equations for the intrinsic manifold are given by

$$-Z \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} + \dot{y} \stackrel{!}{=} 0 \tag{5.8}$$

and hence by

$$[\Theta V - RU + B_0 \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & -\frac{1}{\gamma} \end{pmatrix}]\ [\begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} e^{J\theta}]\ y \stackrel{!}{=} 0. \tag{5.9}$$

This equation just has the solution $y = 0$ since the two matrices in brackets are nonsingular – note that the first one is upper triangular.

Remark:

Even near the equilibria the intrinsic manifold does not approximate the true slow manifold.

For simulations see the corresponding Figures 10– 13.

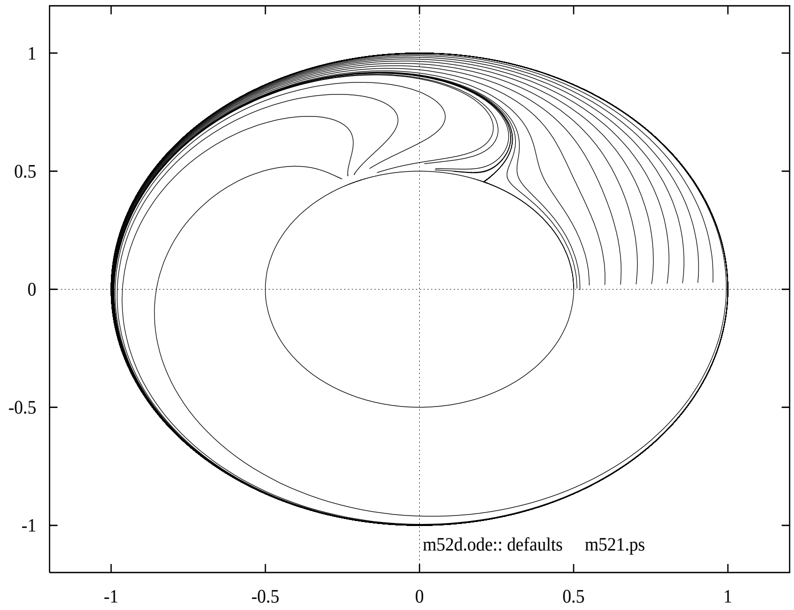

Figure 10: m521.ps

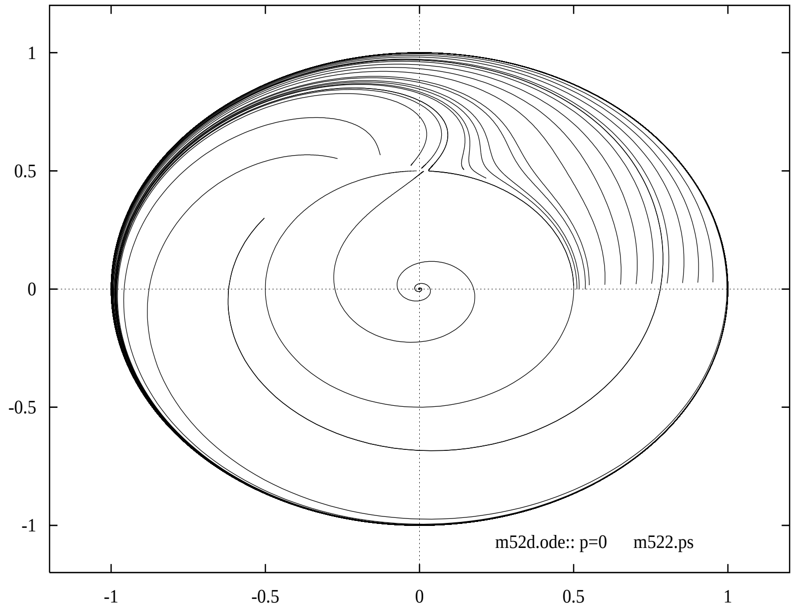

Figure 11: m522.ps

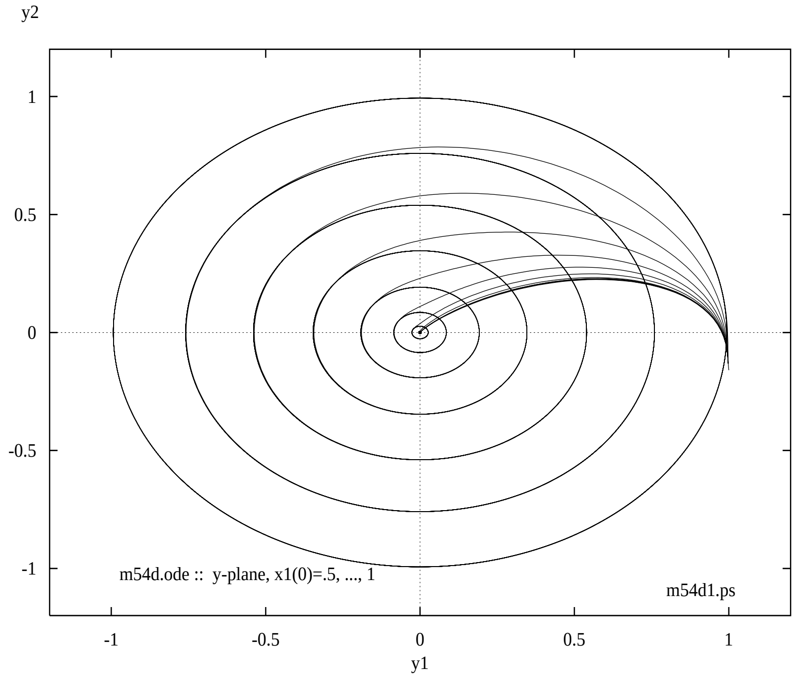

Figure 12: m54d1.ps

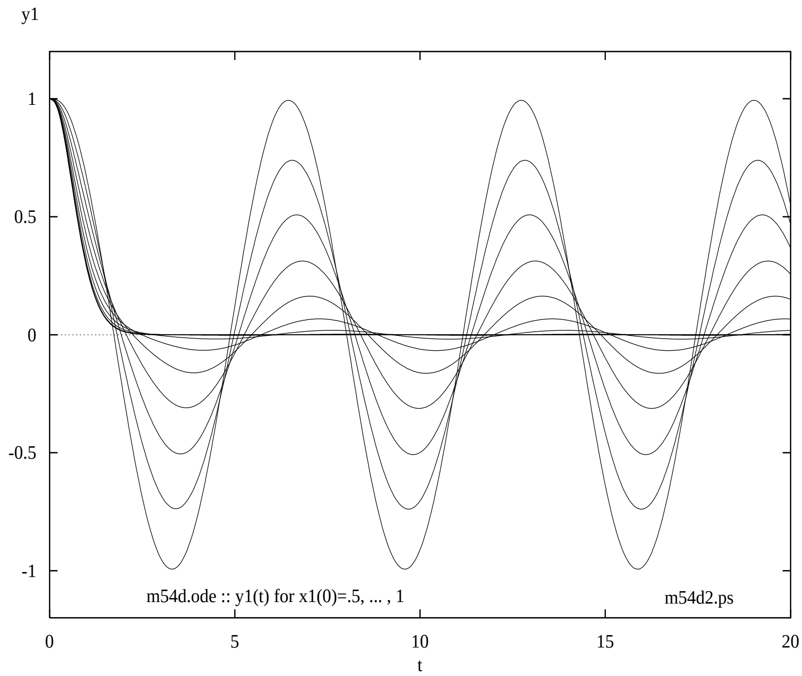

Figure 13: m54d2.ps

6 Appendix

6.1 Fenichel's Reduction

Given the smooth system

$$\begin{aligned} \dot{x} &= f(x,y) + \varepsilon X(x,y,\varepsilon) = F(x,y,\varepsilon), \\ \dot{y} &= g(x,y) + \varepsilon Y(x,y,\varepsilon) = G(x,y,\varepsilon) \end{aligned} \tag{6.10}$$

with a smooth solution $y = u(x)$ of

$$f(x,y) = 0,\ g(x,y) = 0\ . \tag{6.11}$$

Suppose that – with $U = u_x(x)$ –

$$\begin{pmatrix} I \\ U \end{pmatrix} \text{ is a basis of the kernel of } J_0(x) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \text{ along } (x,u(x)). \tag{6.12}$$

Then one has

$$f_x + f_y U \equiv 0, \quad g_x + g_y U \equiv 0 \tag{6.13}$$

along $(x,u(x))$ and the settings

$$B_0 = g_y - U f_y, \quad R = f_y B_0^{-1}, \quad Q_0 = \begin{pmatrix} I & R \\ U & I + UR \end{pmatrix} \tag{6.14}$$

along $(x,u(x))$ lead to the block–diagonalization

$$J_0 Q_0 = Q_0 \begin{pmatrix} 0 & 0 \\ 0 & B_0 \end{pmatrix}\ . \tag{6.15}$$

With the formulas

$$Q_0^{-1} = \begin{pmatrix} I + RU & -R \\ -U & I \end{pmatrix}, \quad P_0 = \begin{pmatrix} I \\ U \end{pmatrix} \begin{pmatrix} I + RU, & -R \end{pmatrix} \tag{6.16}$$

for the inverse of Q_0 and the projection P_0 onto the kernel of J_0 along the complementary eigenspace the reduced equation is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \varepsilon P_0 \begin{pmatrix} X(x,y,0) \\ Y(x,y,0) \end{pmatrix} + \mathcal{O}(\varepsilon^2) \tag{6.17}$$

or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = P_0 \begin{pmatrix} X(x,y,0) \\ Y(x,y,0) \end{pmatrix} \tag{6.18}$$

in slow time $\tau = \varepsilon t$ and in first order ε–approximation. See [2] Lemma 5.4.

In [4] Maas & Pope start with the Jacobian

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} (x,y,\varepsilon) \tag{6.19}$$

and assume a block–diagonalization

$$JQ = Q \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{6.20}$$

with separated spectra as in (1.4). Let us assume $Q = Q(x,y,\varepsilon)$ to be of the form

$$Q = \begin{pmatrix} I & S \\ Z & I + ZS \end{pmatrix} \tag{6.21}$$

with

$$A = F_x + F_y Z, \quad B = G_y - ZF_y \tag{6.22}$$

where Z and S satisfy the Riccati equation and the Liapunov equation

$$G_x + G_y Z = Z(F_x + F_y Z), \quad AS + F_y = SB \tag{6.23}$$

respectively. The projection (6.17) now is

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} I \\ Z \end{pmatrix} \begin{pmatrix} I + SZ, & -S \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} (x,y,\varepsilon) \\ &= \begin{pmatrix} I \\ Z \end{pmatrix} [F + S(ZF - G)](x,y,\varepsilon). \end{aligned} \tag{6.24}$$

Maas & Pope solve the reduction equation (1.6)

$$[ZF - G](x,y,\varepsilon) = 0 \quad \text{for} \;\; y = s(x,\varepsilon) \tag{6.25}$$

(let's say) so that (6.24) becomes

$$\dot{x} = F, \quad \dot{y} = ZF \quad \text{along } (x, s(x,\varepsilon), \varepsilon). \tag{6.26}$$

Note that $y = s(x,0)$ need not be a solution of (6.11) and $s_x(x,\varepsilon)$ need not be such that it could replace $Z(x, s(x,\varepsilon), \varepsilon)$ in (6.21)–(6.26) – not even for $\varepsilon = 0$. Note that along $y = s(x,\varepsilon)$ one should have $\dot{y} = s_x F$ instead of (6.26).

6.2 Remark to Duchêne & Rouchon [1]

We present an example that the reduced system in the papers [1] by Duchêne & Rouchon does not necessarily reflect the dynamics of the original model. We consider smooth systems like

$$\begin{aligned} \dot{x} &= \varepsilon(s_0(x) - y) + \varepsilon^2 h(x), \\ \dot{y} &= s_0(x) + \varepsilon s_1(x) - y + \varepsilon^2[h(x) - s_1(x)][s_0'(x) + \varepsilon s_1'(x)] \end{aligned} \tag{6.27}$$

possessing the slow invariant manifold

$$y = s(x, \varepsilon) \equiv s_0(x) + \varepsilon s_1(x) \tag{6.28}$$

over compact intervals. In the simulation of Figure 14 we take

$$s_0(x) = x^2, \quad s_1(x) = x - 4x^3, \quad h(x) = (1 - \gamma)x. \tag{6.29}$$

The exact reduced system thus is given by

$$\dot{x} = \varepsilon^2(h(x) - s_1(x)) \tag{6.30}$$

whereas the reduced equation on the quasistationary manifold $y = s_0(x)$ in the terminology of [1] is given by

$$\dot{x} \equiv \varepsilon^2 h(x), \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ s_0'(x) \end{pmatrix} \varepsilon^2 h(x). \tag{6.31}$$

The dynamical behavior of (6.30) and the one of (6.31) which – for $\gamma = 0.81$ – now read

$$\dot{x} = \varepsilon^2 x[4x^2 - 0.81] \tag{6.32}$$

and

$$\dot{x} = \varepsilon^2 0.19x \tag{6.33}$$

is not the same. (6.32) has the stable equilibrium $x_0 = 0$ and the unstable equilibria $x_\pm = \pm 0.45$ whereas (6.33) has $x_0 = 0$ as the only (unstable) equilibrium.

An interesting case might be $h(x) = \frac{1}{2}s_1(x)$ with s_0, s_1 as in (6.29) leading to time reversed dynamics in the reduced equations (6.30) and (6.31) resp.

A little more general:

$$\begin{aligned} \dot{x} &= \varepsilon[s_0(x) + X(x, y, \varepsilon) - y], \\ \dot{y} &= s_0(x) + \varepsilon s_1(x, \varepsilon) - y + \varepsilon Y(x, y, \varepsilon) \end{aligned} \tag{6.34}$$

has the exact slow manifold $y = s_0(x) + \varepsilon s_1(x, \varepsilon) \equiv s(x, \varepsilon)$ over compact intervals for

$$Y(x, s(x, \varepsilon), \varepsilon) = s_x(x, \varepsilon)[X(x, s(x, \varepsilon), \varepsilon) - \varepsilon s_1(x, \varepsilon)]. \tag{6.35}$$

Its reduced flow is governed by

$$\dot{x} = \varepsilon[X(x, s(x, \varepsilon), \varepsilon) - \varepsilon s_1(x, \varepsilon)], \tag{6.36}$$

whereas the flow on the quasistationary manifold $y = s_0(x)$ is determined by $\dot{x} = \varepsilon X(x, s_0(x), 0)$. It is in general not a reliable approximation of (6.36).

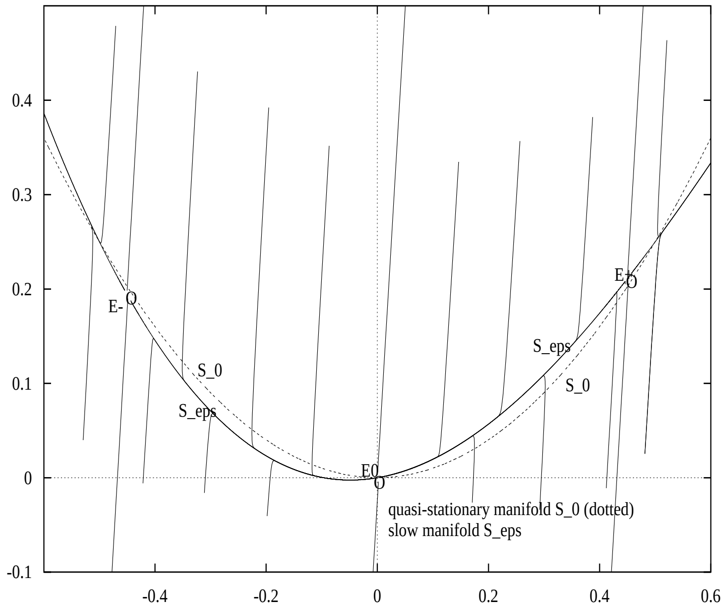

Figure 14:
System (6.27) with (6.29) and $\varepsilon = .1$, $\gamma = .81$ and quasistationary manifold $y = x^2$ and slow manifold $y = x^2 + \varepsilon[x - 4x^3]$

Remark:
Given a system

$$\varepsilon \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} M \\ I \end{pmatrix} \omega(x, y) + \varepsilon \begin{pmatrix} X \\ Y \end{pmatrix} \tag{6.37}$$

with a constant matrix M as in [1] the transformation

$$u = x - My \tag{6.38}$$

leads to

$$\varepsilon \begin{pmatrix} \dot{u} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega(u + My, y) \end{pmatrix} + \varepsilon \begin{pmatrix} X - MY \\ Y \end{pmatrix} (u + My, y). \tag{6.39}$$

Suppose now that

$$y = \sigma(u) = s(x) \quad \text{with} \quad s_x \equiv -Z \equiv -\omega_y^{-1}\omega_x \tag{6.40}$$

is <u>the</u> solution of

$$\omega(u + My, y) = \omega(x, y) \equiv 0.$$

Then $s(u + M\sigma(u)) = \sigma(u)$ implies

$$\sigma_u = -(I + ZM)^{-1}Z \tag{6.41}$$

and – by $(I - M(I + ZM)^{-1}Z) = (I + MZ)^{-1}$ –

$$\begin{aligned} \varepsilon\dot{x} &= \varepsilon(I + M\sigma_u)\dot{u} \\ &= \varepsilon(I - M(I + ZM)^{-1}Z)(I, -M)\begin{pmatrix} X \\ Y \end{pmatrix} \\ &= \varepsilon(I + MZ)^{-1}(I, -M)\begin{pmatrix} X \\ Y \end{pmatrix}. \end{aligned} \tag{6.42}$$

Hence the *reduced(?)* system can be written as

$$\varepsilon\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \varepsilon\begin{pmatrix} I \\ -Z \end{pmatrix}(I + MZ)^{-1}(I, -M)\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{6.43}$$

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