MAX–PLANCK–INSTITUT
DYNAMIK KOMPLEXER
TECHNISCHER SYSTEME
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Nonlinear Systems – WS 2008/09
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Exercise 4.0 – Hartman–Grobman Homeomorphisms

Consider the linear initial value problems

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x, \quad x(0) = \xi, \tag{4.0.1}$$

and

$$\dot{y} = \begin{pmatrix} -1 & \omega \\ -\omega & -1 \end{pmatrix} y, \quad y(0) = \eta, \tag{4.0.2}$$

and

$$\dot{z} = \begin{pmatrix} -1 & \omega \\ 0 & -1 \end{pmatrix} z, \quad z(0) = \zeta, \tag{4.0.3}$$

with (small) $\omega > 0$ and denote their solutions by $x(t,\xi)$, $y(t,\eta)$ and $z(t,\zeta)$ respectively. Determine two homeomorphisms G and H (on neighborhoods of the respective origins) with

$$G(x(t,\xi)) = y(t, G(\xi)) \quad \text{and} \quad H(x(t,\xi)) = z(t, H(\xi)). \tag{4.0.4}$$

Hint: Over $\mathbb{C}$, (4.0.1) and (4.0.2) read $\dot{x}_1 + i\dot{x}_2 = -(x_1 + ix_2)$ and $\dot{y}_1 + i\dot{y}_2 = -(1+i\omega)(y_1 + iy_2)$. Thus G should *somehow correspond* to raising to the power $(1 + i\omega)$.
So, Ex.3.3 is really an exercise about taking complex powers of complex variables. ■

Exercise 4.1 – from Chapter 2 of the handout

Do the exercises called *Aufgabe 2.4*, *Aufgabe 2.5*, *Aufgabe 2.7*, *Aufgabe 2.12*, *Aufgabe 2.14*. ■

Exercise 4.2 – Hamiltonian Systems (see Chapter 2 of the handout)

a1) Produce the phase portraits of

$$\dot{u} = H_v(u,v), \ \dot{v} = -H_u(u,v) \quad \text{for } H(u,v) = \tfrac{1}{2}[v^2 - u^2] + \tfrac{1}{4}u^4 \tag{4.2.1}$$

numerically and analytically.

a2) With small $\alpha > 0$, consider the perturbations of (4.2.1) given by

$$\dot{u} = v, \quad \dot{v} = u - u^3 - \alpha v \tag{4.2.2}$$

$$\dot{u} = v - \alpha H_u H, \quad \dot{v} = u - u^3 - \alpha H_v H \tag{4.2.3}$$

$$\dot{u} = v, \quad \dot{v} = u - u^3 \pm \alpha v H(u, v). \tag{4.2.4}$$

and produce their phase portraits numerically.

b) Generate the phase portraits of

$$\dot{u} = K_v(u, v), \ \dot{v} = -K_u(u, v) \quad \text{for } K(u, v) = \tfrac{1}{6}(u - v)^3 - uv, \tag{4.2.5}$$

numerically and analytically. ■

Excursion 4.3 – Flow Box Theorem and First Integrals

(a) A C^1–system $\dot{x} = f(x)$ in $\mathbb{R}^n$ with $f(0) = e_1$ (and with $\phi^t(x_0)$ denoting its solution through x_0 at time $t = 0$) is locally equivalent to $\dot{y} = e_1$. Show that the mapping

$$y = (s, \xi_2, ...\xi_n)^{\mathrm{T}} \mapsto x = H(y) := \phi(s, (0, \xi_2, ...\xi_n)^{\mathrm{T}}) \tag{4.3.1}$$

satisfies $DH(0) = id$. By the Inverse Function Theorem, H is a local diffeomorphism. Show that the local change $y = H^{-1}(x)$ of coordinates induces $\dot{y} = e_1$.
Verify $H \circ \psi^t = \phi^t \circ H$ (with $\psi^t(y_0)$ denoting the solution of $\dot{y} = e_1$ through y_0 at time $t = 0$).

(b) Verify that first integrals for $\dot{x} = f(x)$ near 0 are given by

$$I_j(x) = e_j^{\mathrm{T}} H^{-1}(x) \, (= \xi_j), \quad j = 2, ..., n.$$

and that, for any smooth $G \in C^1$, the following function is a first integral again:

$$I(x) = G(I_1(x), ..., I_{n-1}(x)).$$

(c) Consider the affine system

$$\dot{u} = 1, \quad \dot{x} = -3u, \quad \dot{y} = -x \tag{4.3.2}$$

and compute a coordinate transformation $(s, \xi, \eta)) \mapsto (u, x, y) = H(s, \xi, \eta)$ near the origin in such a way that the resulting ODE is

$$\dot{s} = 1, \quad \dot{\xi} = 0, \quad \dot{\eta} = 0 \tag{4.3.3}$$

with ξ and η being first integrals.

(d) Consider (4.3.2) as being the ODE-system of a first-order PDE for $u = u(x, y)$ or $x = x(u, y)$ or $y = y(u, x)$. Derive the corresponding PDEs. Suppose, furthermore, there is a given initial curve $(u, x, y) = (0, x, A(x))$ for x in a neighborhood of 0. Compute the solution of the initial value problem for the PDE. HINT: Use the representation $y = y(u, x)$. ■

Exercise 4.4 – Simulations for 'Takens-Bogdanov'

For smooth 2–dimensional sytems $\dot{x} = F(x, p_1, p_2)$ depending on two parameters with

$$F(0,0,0) = 0, \quad F_x(0,0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the Takens–Bogdanov normal form is given by (4.5.12) below. Here, consider the planar time-invariant differential system

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = a + bx_1 + x_1^2 - x_1x_2 \tag{4.4.1}$$

with real parameters a, b. Compute the equilibria $E_\pm = E_\pm(a,b)$ and and consider the linearizations of (4.4.1) at $E_\pm(a,b)$. Decide for what (a,b) they correspond to saddles, to stable nodes (spirales) or to unstable nodes (spirales).

Run simulations and generate numerically phase portraits of (4.4.1) for

$$a = cos(s), \;\; b = -sin(s)$$

for a seqeunce of s-values running from 0 to 2π. What phenomena can be observed over the s-interval $[0.49 \cdot \pi, 0.57 \cdot \pi]$? ■

Exercise 4.5 – Normal Form Computations

Consider the planar time-invariant differential system

$$\dot{x} = Ax + f(x) \tag{4.5.1}$$

where f belongs to the class $\mathcal{P}$ of homogeneous polynomials of degree $d = 2$:

$$f(x) = \textstyle\sum_{i=1}^{6} b_i(x)\beta_i \equiv B(x)\beta \tag{4.5.2}$$

with respect to basis-vectors

$$\begin{aligned} b_1(x) &= \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \; b_2(x) = \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \; b_3(x) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \\ b_4(x) &= \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \; b_5(x) = \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \; b_6(x) = \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} \end{aligned} \tag{4.5.3}$$

and $B(x) = [b_1(x), ..., b_6(x)]$ and $\beta = (\beta_1, ..., \beta_6)^{\mathrm{T}}$. The GOAL is to find a local change of coordinates

$$y = x + h(x) \quad \text{with} \quad h \in \mathcal{P} \tag{4.5.4}$$

(with inverse $x \stackrel{?}{=} y - h(y) + \mathcal{O}(|y|^3)$) such that in the resulting transformed differential equation

$$\dot{y} = Ay + q(y), \quad q(y) \stackrel{!}{=} \mathcal{O}(|y|^3), \tag{4.5.5}$$

the nonlinearity q does not contain any quadratic terms.

(a) When is this possible for a diagonal $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$?

(b) When is this possible for $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \in \mathbb{R}^{2\times 2}$?

<u>Hint:</u> That's really an exercise in Linear Algebra. With the Jacobian Dh of h and with

$$\mathcal{L}_A(h)(y) := Dh(y)Ay - Ah(y) \tag{4.5.6}$$

the transformed equation reads

$$\dot{y} = Ay + \ [\mathcal{L}_A(h)(y) + f(y)] \ + \mathcal{O}(|y|^3). \tag{4.5.7}$$

$\mathcal{L}_A$ is a linear operator acting on $\mathcal{P}$. Compute $\mathcal{L}_A(b_i)$ for the basis elements from (4.5.3) and derive a (6×6)–matrix representation L_A for $\mathcal{L}_A$ (with $\mathcal{L}_A B = BL_A$). What are the eigenvalues of $\mathcal{L}_A$? Under what conditions can $\mathcal{L}_A(h)(y) + f(y) \overset{!}{=} 0$ be solved for $h \in \mathcal{P}$ for all $f \in \mathcal{P}$? ■

Excursion 4.6 – Normal Form Computations (Exercise 4.5 cont.)

(c) For A of the form $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ with $\beta \neq 0$ complex notations are used: With $\lambda = \alpha + i\beta$ and

$$\begin{aligned} \dot{z} &= \lambda z + F_{20}z^2 + F_{11}z\bar{z} + F_{02}\bar{z}^2 \quad \text{(for (4.5.1))}, \\ w &= z + H_{20}z^2 + H_{11}z\bar{z} + H_{02}\bar{z}^2 \quad \text{(for (4.5.4))} \end{aligned} \tag{4.5.8}$$

one proceeds in a completely analogous way.

<u>Answer:</u> In all 3 cases (a), (b) and (c) the conditions are

$$(m_1\lambda_1 + m_2\lambda_2) - \lambda_j \neq 0 \tag{4.5.9}$$

for $m_1.m_2 \in \mathbb{N}_0$ with $m_1 + m_2 = d = 2$ and the eigenvalues λ_j of A $(j = 1, 2)$.
Otherwise, some of the 6 quadratic terms in (4.5.2) cannot be eliminated.

(d) Let's look at part (b) with $\lambda = 0$ where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case L_A has rank 4. So there exist 2–dimensional subspaces $\mathcal{S}$ complementary to the 4–dimensional range $\mathcal{R}(\mathcal{L}_A)$. Show that

$$span\{b_1, b_4\} \quad \text{and} \quad span\{b_4, b_5\} \tag{4.5.10}$$

are two possible choices for $\mathcal{S}$. Hence one can assume w.l.o.g. (4.5.1) to be of the form

$$\begin{aligned} &(I) \quad \dot{x}_1 = x_2 + ax_1^2 + \mathcal{O}(|x|^3)\,, \quad \dot{x}_2 = bx_1^2 + \mathcal{O}(|x|^3) \\ &\quad\quad\ \text{or} \\ &(II) \quad \dot{x}_1 = x_2 + \mathcal{O}(|x|^3)\,, \qquad\quad \dot{x}_2 = cx_1^2 + dx_1x_2 + \mathcal{O}(|x|^3) \end{aligned} \tag{4.5.11}$$

respectively with just 2 quadratic terms. By an approptiate scaling the coefficients may be taken out of $\{0, \pm 1\}$.

(e) TAKENS–BOGDANOV NORMALFORM:
For smooth 2–dimensional sytems $\dot{x} = F(x, p_1, p_2)$ depending on two real parameters with

$$F(0,0,0) = 0, \quad F_x(0,0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the Takens–Bogdanov normal form is given by

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = a + bx_1 + x_1^2 \pm x_1 x_2 + \mathcal{O}(|x|^3) \tag{4.5.12}$$

with two (new) real parameters a and b. Lit.: Kuznetsov p.321 (Springer Appl.Math.Sci.112). ■